ROOT-LOCI PLOT BY DIGITAL COMPUTER

T. L. Chandranath, C. V. R. Wirasinghe and J. B. X. Devotta

Synopsis

The development of a general computer program for the accurate determination of the Root-Loci plot of Control Systems is described. Algorithms are developed based on Routh's Criterion for the imaginary axis cross-over frequencies and the value of loop gain at the cross-over points of the locus. The entire loci plot is obtained by successive shifts of the imaginary axis of the s-plane. The computer program developed is designed to give results in the form of a graphical display as well as a numerical tabular form.

1. Introduction

The main task of the control system designer is to ensure that the system response meets the specifications. To do this he should be able to predict the system response, from a knowledge of the system parameters and the system dynamics. The time response of a control system in particular, is determined by the location of the poles and zeros of the loop transfer function, and the loop gain. Of the several methods available for the prediction of system response, the Root Locus approach of Evans[1] has several advantages. This method combines the theoretical advantages of simultaneous control of both the transient response and frequency response of the system, and provides correlation between the system parameters and system response[2]. An approximate sketch of the Root Loci plot can be obtained using the set of elementary rules given in literature on Control Systems.[1,2,3,4] However an accurate plot can be obtained only by locating several points of the locus by a trial and error method. Alternatively algorithms can be derived for a point by point development of the Root Loci plot. The vast amount of effort involved in the numerical computation of this method, is greatly simplified by the use of the digital computer.

2. Theoretical Development

The transfer function of a linear feedback control system can be written as

$$GH(s) = \frac{K\sum_{0}^{m} b_i s^i}{\sum_{0}^{n} a_i s^i} \qquad m \leqslant n \qquad \ldots\ldots\ldots (1)$$

The system characteristic equation, $1 + GH(s) = 0$ becomes

$$\sum_{0}^{n} a_i s^i + K\sum_{0}^{m} b_i s^i = 0 \qquad \ldots\ldots\ldots (2)$$

The Root Locus is defined as the locus of points in the s-plane which satisfy equation (2), as K is allowed to take any real value. (Usually we consider only the range $0 < K < \infty$, but negative values of K can also be considered if so desired).

By grouping terms of similar powers of s, equation (2) can be written as

$$\sum_{0}^{n} c_i s^i = 0, \text{ where } c_i = \phi(K) \text{ in general.} \qquad \ldots\ldots\ldots (3)$$

If the origin of the s plane is a point of the Root Locus then, from (2), the value of K at this point is given

$$\text{by } K_o = -\left(\frac{a_o}{b_o}\right) \qquad \ldots\ldots\ldots (4)$$

Now consider the application of Routh's Criterion[2,4,5] to equation (2). Writing the Routh Array of coefficients in double suffix notation,

$$\begin{array}{l|l}
s^n & A_{11} \; A_{12} \; A_{13} \; \ldots\ldots A_{1p} \\
s^{n-1} & A_{21} \; A_{22} \; A_{23} \; \ldots\ldots \\
s^{n-2} & A_{31} \; \ldots\ldots \\
\ldots & \ldots\ldots \\
s^2 & A_{n-1,1} A_{n-1,2} \; \ldots\ldots \\
s^1 & A_{n,1} \; 0 \\
s^0 & A_{n+1,1} \; 0
\end{array}$$

where, for $j \leqslant 2$,

$$A_{1,k} = c_{n-2,k+2}$$

$$A_{2,k} = c_{n-2,k+1}$$

and for $j > 2$,

$$A_{j,k} = A_{j-2,k+1} - \left[\frac{A_{j-2,1}\,A_{j-1,k+1}}{A_{j-1,1}}\right] \ldots (5)$$

J. B. X. Devotta, BSc (Eng), PhD graduated from the University of Sri Lanka in 1965 and joined the staff of the University in the same year. In September 1969, he joined the Power Systems group of the Imperial College of Science and Technology as a research student. His work was on System Stability with special reference to the DWR Machine and included both experimental and theoretical work on the Micromachine installations and digital computer respectively. He returned to Sri Lanka in 1973 and rejoined the University Staff.

T. L. Chandranath and C. V. R. Wirasinghe are final-year Electrical Engineering students of the University of Sri Lanka, Peradeniya Campus.

Since all c_i coefficients are known, all $A_{j,k}$ coefficients can be evaluated using the algorithm given above.

Now if for some value of K equation (3) contains a factor $(s^2+w_0^2)$ it can be shown[5] that the necessary and sufficient conditions are given by

$$A_{n,1}(K) = O \qquad \text{..............(6)}$$

and $$\frac{A_{n-1,2}(K)}{A_{n-1,1}(K)} = w_0^2 \qquad \text{..............(7)}$$

The significance of equations (6) and (7) is that for a value of K satisfying equation (6), the Root Locus has a point on the imaginary axis at a value w given by equation (7). Thus the point of intersection of the Root Locus with the imaginary axis (i.e. the w axis) is known.

Now consider the linear transformation of equation (2) by the relation $s_0 = s + r$. This is equivalent to shifting the imaginary axis of the s plane to the right by an amount r. Equation (2) now becomes, in terms of the new variable s_0,

$$\sum_{0}^{n} a_i{}' s_0{}^i + K \sum_{0}^{m} b_i{}' s_0{}^i = O \qquad \text{......... (8)}$$

The coefficients a_i' and b_i' are functions of the shift r and of the order of the polynomials n and m. These can be evaluated easily using the Binomial series expansion. In the Computer Program these coefficients are computed using the subroutine PCLD (Polynomial Complete Linear Synthetic Division) supplied by the IBM 1130 Scientific Subroutine Package.

Since the shape of the Root Locus plot is unaffected by the linear transformation[5,6], we can apply relations (6) and (7) to the transformed equation (8), to find the crossover on the new (shifted) imaginary axis and the corresponding value of K. The value of K at the new origin is given as before by equation (4) i.e. $K = -(\frac{a_0{}'}{b_0{}'})$. This method is illustrated by the following example.

3. Illustrative Example

Consider the 4th order system of loop transfer function—

$$GH(s) = \frac{K(s+2)}{s^2(s+5)(s+10)}$$

The characteristic equation is—

$$s^4 + 15s^3 + 50s^2 + Ks + 2K = 0.$$

Equations (6) and (7) become

$$A_{n,1} = K(750 - K) = 0$$

and $$w_0^2 = \frac{30K}{(750 - K)}$$

The solution of these equations give

$$\left.\begin{matrix} K = 0 \\ w_0 = 0 \end{matrix}\right\} \text{ and } \left.\begin{matrix} K = 750 \\ w_0 = \pm\sqrt{20} \end{matrix}\right\}$$

Suppose we shift the imaginary axis to the left by unity i.e. $s_0 = s - 1$. This transformation gives the new characteristic equation as—

$$s_0{}^4 + 17s_0{}^3 + 50s_0{}^2 + (82 + K)s_0 + (35 + K) = O,$$

and equations (6) and (7) become

$$A_{n,1} = (768 - K)(82+K) - (17 \times 17)(35+K) = 0 \quad \text{...(9)}$$

and $$w_0{}^2 = \frac{17(35 + K)}{768 - K} \qquad \text{......................(10)}$$

K and w_0 can be found from these two equations. Thus the problem of locating the zeros fourth order system characteristic equation is now reduced to a second order equation (9).

It should be noted that both negative values for w_0^2 and imaginary values for K, if obtained from equations (9) and (10), are meaningless as far as the Root Loci plot is concerned.

4. The Computer Program

The flowchart for the general method of applying the above theory is given in Figure 2. There are two loops involved, of which the outer loop performs the repetitive shifting of the imaginary axis, and the inner loop deals with finding the zeros of equation (6). Since equation (6) contains K in an implicit form, some form of numerical analytical method has to be used to solve it. One method could be the determination of the sign of $A_{n,1}$ as K is varied from 0 to ∞ in steps, and when a sign change occurs a more accurate estimate for K can be got by progressive subdivision or Newton's method. It should be noted that the computation time for this method can be very large, and also the roots may be missed altogether if they lie very close to one another, comparable to the increment ΔK used. A more direct approach would be to rearrange equation (6) in an explicit form. We wish to find the zeros of $A_{n,1} = 0$. But

$$A_{n,1} = F(a_1', a_2', \ldots\ldots a_n', b_1', b_2' \ldots\ldots b_m', K) = O$$

An explicit solution for K would be

$$K = \phi(a_1', a_2', \ldots\ldots a_n', b_1', b_2' \ldots\ldots b_m')$$

Algorithms were developed for determining these functions ϕ.

It was found that the algebraic task involved in finding these functions ϕ, becomes very complicated when the combined order $(m + n)$ becomes large. In the program developed these algorithms were restricted to the range $(m+n) \leqslant 6$ (see Figure 2) i.e. the sum of the number of poles and zeros of the loop transfer function do not exceed 6. An example of the derivation of the algorithm is given in the Appendix.

The limitation $(m+n) \leqslant 6$, is not very severe, since in most practical problems the number of poles seldom exceed 3 or 4 and the number of zeros seldom exceed 2 or 3. In more complex cases, by judicious approximations, the system can be reduced to the desired form. Even in cases where such simplification is not desirable, we can apply the general method explained earlier in this section.

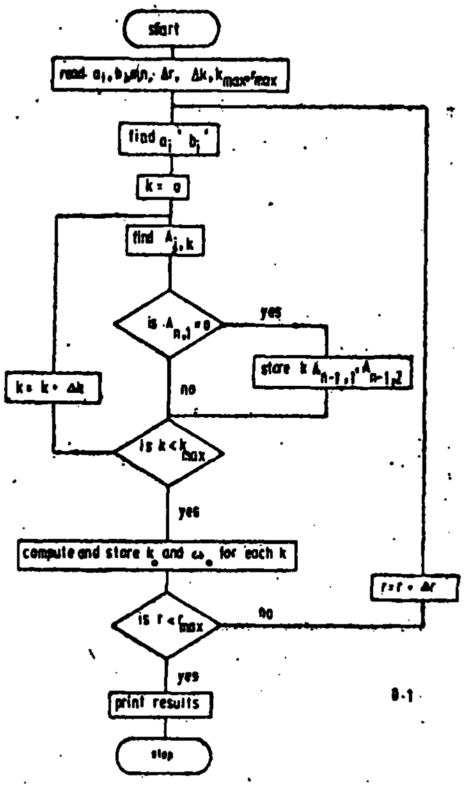

Fig. 1. Flow Chart for the general method

POLES \ ZEROS	0	1	2	3	4	5	6
0	T						
1	T	T					
2	•	•	•				
3	•	•	•	•			
4	•	•	•	N	N		
5	•	•	N	N	N	N	
6	•	N	N	N	N	N	

• Algorithms developed
T Trivial cases
N Algorithms not developed

D-2

Fig. 2. Pole-Zero Combinations of GH (s)

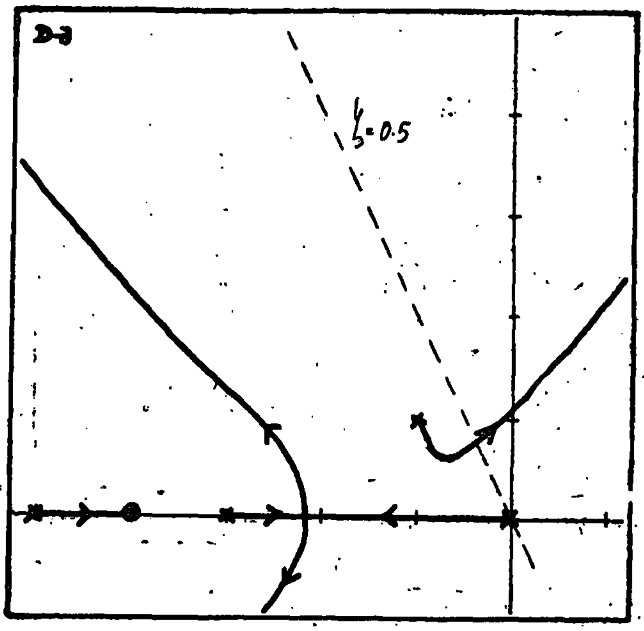

Fig. 3. Root Locus plot of Ex. (1)

The computer program RTLUS is designed to give the output in a tabular form giving the values of the shift r, the value of K on the real axis branch of the Locus, the crossover frequencies (there can be more than one) and the corresponding values of K at these points. The user has to specify the range of s plane to be covered and the poles and zeros of the loop transfer function. Optional facilities include a graphical display of the Root Locus plot by using the IBM 1627 Plotter. A diagnostic message NOK will be printed if by chance (i) the shifting of the axis coincides with an open loop zero or when (ii) the Root Loci plot contains a branch parallel to the imaginary axis. (In the former case the value of K becomes infinity and in the latter case w_0 is indefinite).

5. Examples

Ex. 1 : Suppose that the loop transfer function of a system, GH(s) is $\dfrac{4(s+4)}{s(s+3)(s+5)(s^2+2s+2)}$, and we wish to select a suitable value for Loop Gain and also locate the system poles for the chosen conditions. The Root Locus plot as obtained from the computer is shown in Fig. 3. The output also shows that the system is stable for values of K between 0 and 10.6. It also reveals that for a damping ratio of 0.5, K should be about 3.8. For this value of K the closed loop system poles are at

$$0.4 \pm j0.76, \ -1.37, \ -4.98 \text{ and } -2.85.$$

The impulse response of the closed loop system will be highly influenced by the pair of complex roots and the roots at $s = -1.37$ and $s = -2.85$. The contribution from the remaining root is very small.

Ex. 2 : The Root Locus plot for

$$GH(s) = \frac{K(s+8)(s+3.25)}{s(s+3)(s+6)} \text{ is shown in Fig. 4.}$$

Note that the exact shape of this plot, especially the pear-shaped portion of the complex branches, cannot be obtained by using the Evan's rules for construction of the Root Loci. Observe also that for a wide range of K, from about 3.0 to 10, the damping ratio is about 0.88, and that the minimum value of damping ratio possible is 0.86.

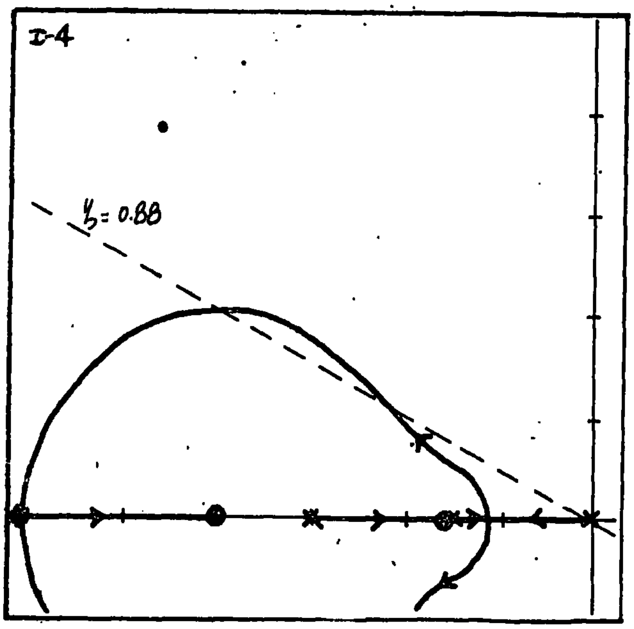

Fig. 4. Root Locus plot of Ex. 2

Ex. 3: The loop transfer function of autoplot system of a rigid missile is known to be

$$\frac{K(s+0.075)}{(s+2.67)(s-2.57)(s-0.0105)(s^2+14s+100)}$$

It is desired to stabilise the system by providing viscous damping of the tail surface. In effect, this introduces an additional term of the type $(1+\frac{k_1}{k_2}s)$ in the loop transfer function. The Root Locus plot of the uncompensated system is shown in Fig. 5a. Note that the system is unstable for all K. Selecting $\frac{k_1}{k_2}$ as 2.67, (i.e. cancellation of the dominant non-zero loop pole) the Root Locus of the compensated system is shown in Fig. 5b. This shows a conditionally stable system, for effective K between 261 and 973.

6. References

1. Evans, W. R., '*A Graphical Analysis of Control Systems*', Trans. AIEE, Vol. 67, pp. 547-551.
2. Truxal, J. G., '*Automatic Feedback Control System Synthesis*', McGraw Hill, 1955.
3. Grabbe, Ram and Woolridge, '*Handbook of Automation Computation and Control*', Vol. 1 (Book), John Wiley, 1965.
4. D'Azzo and Houpis, '*Feedback Control Systems Analysis and Synthesis*' (Book), McGraw Hill, 2nd Ed.
5. Chen and Hass, '*Elements of Control Systems Analysis*', Prentice Hall, 1969.
6. Bendrikov and Theodorick, '*A Simplified Method of Analytical Construction of Root Trajectories*', Automation i Telemekhanika, Vol. 24, No. 2, Feb. 1963, pp. 268-270.

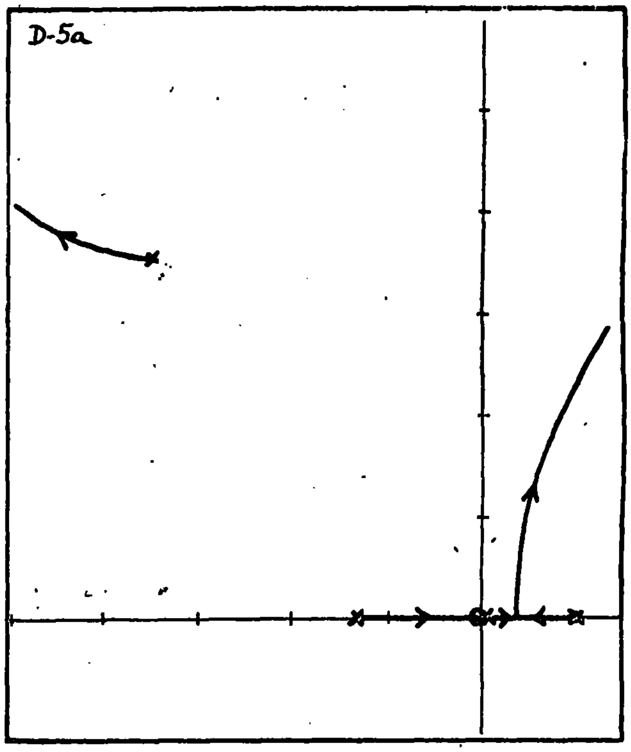

Fig. 5a. Root Locus Plot of Ex. 3
(uncompensated System)

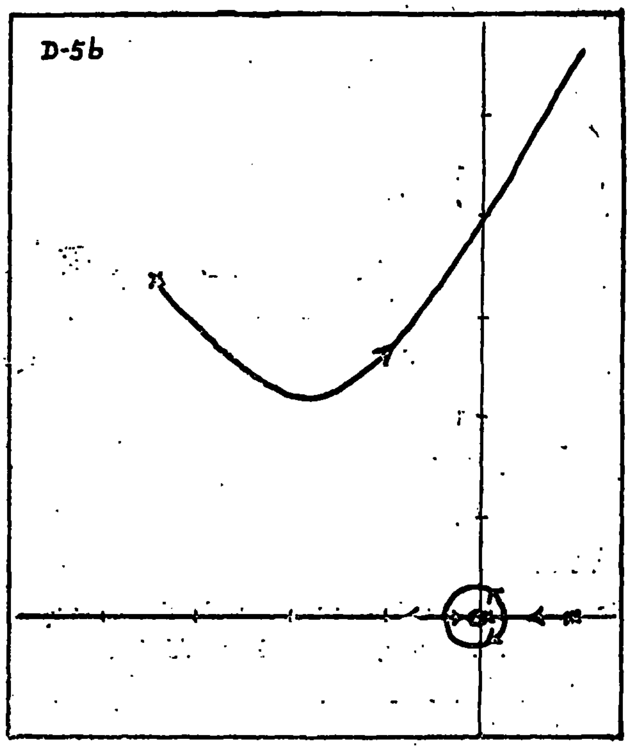

Fig. 5b. Root Locus Plot of Ex. 3
(compensated System)

Appendix

Consider a loop transfer function

$$GH(s) = \frac{K(s^2 + b_1 s + b_0)}{s^3 + a_2 s^2 + a_1 s + a_0}$$

The characteristic equation is

$$s^3 + (a_2 + K)s^2 + (a_1 + b_1 K)s + (a_0 + b_0 K) = 0$$

i.e. $$\sum_{0}^{3} c_i s^i = 0$$

The Routh Array is

s^3	c_3	c_1	
s^2	c_2	c_0	
s'	$c_1 - \left(\frac{c_0 c_3}{c_2}\right)$	0	
s^0	c_0		

Equations (6) and (7) become,

$$A_{n,1} = c_1 - \frac{c_0 c_3}{c_2} = 0, \text{ and } w_0^2 = \frac{A_{n-1,2}}{A_{n-1,1}} = \frac{c_0}{c_2}$$

i.e. $(a_2 + K)(a_1 + b_1 K) = (a_0 + b_0 K)$

and $w_0^2 = (a_1 + b_1 K)$

Solving for w_0^2 and K give,

$$w_0^2 = \frac{1}{2}\left[(b_0 + a_1 - b_1 a_2) \pm \sqrt{(b_0 + a_1 - b_1 a_2)^2 + (a_0 b_1 - a_1 b_0)}\right]$$

$$K = \frac{w_0^2 - a_1}{b_1}$$

ERRATA

Control System Dynamic by Digital Computer by J. B. X. Devotta, in the Dec. 76 Issue.

In Appendix A, the line after equation A, should read

"Also for complex p_i, $(p_i + \overline{p_i}) = 2\mathrm{Re}(p_i)$"

In Appendix B, page 47, the second line from top should be " $f'(-6) =$ "

Library (*Continued from page 36*)

1150 The Challenge of Space, G. K. C. Pardoe.

1151 Tools for Progress, Intermediate Technology Development Group Ltd.

1152 Tools for Progress, Intermediate Technology Development Group Ltd.

1153 Port Dues, Charges and Accommodation, Captain F. S. Campbell.

1154 Encyclopaedia of Factories, Shops and Office Law and Practice, Peter Allsop.

1155 Adjustment and Testing of Radio Equipment, D. Osher.

1156 Quantitative Methods in Construction Management, J. J. Adriah.

1157 Composite Materials (EMSS), L. Holliday.

1158 Building with Large Prefabricates, B. Lewicki.

1159 Creep of Concrete: Plain, Reinforced and Prestressed, A. M. Naville.

1160 Simplified Calculation Methods of Shells, Structures. 1st repr., Paduart A. and B. Datron.

1161 Plastic Analysis and Design of Plates, Shells and Discs, M. A. Save.

1162 A History of Landscape Architecture, G. B. Tobey.

1163 Floor Research on Blading, L. S... Dzung.

1164 Project Surveying, P. Richards.

1165 Counter Current Separation Processes, H. C. R. Pratt.

1166 Computer Simulation Techniques in Hydrology Resources, G. Fleming.

1167 Fluid Mechanics of Internlaw, G. Sevran.

1168 An Introduction to Fuel Cells, E. R. Williams.

1169 Handbook of Cane Sugar Engineering, E. Hugot.

1170 Introduction to Cane Sugar Technology, G. H. Jenkins.

1171 The Mechanics of Crushing Sugar Cane, G. R. Murray and J. E. Holt.

1172 Electrical Conduction in Their Metal Films, T. J. Coutts.

1173 Data Transmission, M. D. Bacon and G. M. Bull.

1174 Simulation Programming Languages (IFIP), J. N. Buxton.

1175 Informal Introduction to Algol 68 (IFIP), C. H. Lindsey and S. G. Vander Meulen.

1176 Handbook on the Unified Code for Structural Concrete (CP. 110 : 1972), Cement Concrete Association.

1177 Code of Practice for the Structural Use of Concrete, B.S.I.

NOTE: Acces. No. 1079 to 1154 — Donated by the British Council.
Acces. No. 1155 to 1177 — Gifted by the Netherlands Alumni Association of Sri Lanka.