RING FINITE ELEMENT BASED ON ASSUMED STRAIN FUNCTIONS

by
Dr. M.C.M. Fonseka
BSc(Eng), PhD(Southampton), AMIE(SL)
Department of Civil Engineering, University of Moratuwa

Abstract

In carrying out linear analysis of geometrically axisymmetric shells of revolution using the Finite Element Method, it is convenient and efficient to use ring elements. This paper discusses a Finite Element model based on assumed strain functions applied to a ring element which can be used in axisymmetric analysis of shells of revolution. In this derivation, the strains are initially assumed by suitable functions and displacement fields are subsequently derived based on these strains, which is different from the classical approach where displacements fields are assumed initially. Previous investigations have revealed that finite element models derived using the strain based approach have a faster rate of convergence than classical models and they are highly competitive with models developed using higher order elements.

1.0 Introduction

1.1 Elements used in shell analysis

Different researchers have developed various types of elements for shell analysis and these elements have been subjected to continuous improvements by later researchers. Basically, these elements can be identified as falling into two groups, flat elements and curved elements[1]. In analysis of shells of revolution, Ring Elements are very useful since modelling of the structure becomes very much simplified and computer time required is reduced. Ring elements can either be flat or curved depending on the application.

Grafton & Strome[2] have used ring elements in applying direct stiffness method to analyse shells of revolution. In this case, the concept of nodal circles is introduced instead of nodal points in normal elements. The prime advantage here is that the condition of rotational symmetry of the shell is taken into account thus giving a reduction in the size of the matrices to be manipulated. The concept of ring elements was again used by Percy et al[3] in their research. However, Grafton & Strome and Percy et al have used conical frustrum which is the flat version of the ring element and this gave rise to similar problems encountered with normal flat elements. Stricklin et al[4] and Jones et al[5] have used a curved version of the ring element. In shell analysis it is pertinent to use curved ring elements with actual shell parameters, since shell behaviour is very sensitive to its geometry.

1.2 Strain based Finite Element model

Much investigation had been carried out to develop techniques to achieve rapid convergence of finite element models. One recourse is to use higher order elements, that is elements with more degrees of freedom. Although higher order elements give a better convergence, the band width of the structure stiffness matrix is larger than a normal element thus giving an increase in the solution time because the solution time is proportional to the square of the band width. Moreover, deriving the element stiffness matrix becomes much involved in higher order elements since complex shape functions are encountered. Considering the above shortcomings in the higher order elements, the recent research trend is oriented towards new approaches and consequently the concept of strain based finite elements was highlighted.

In strain based finite element models, element shape functions are derived using simple independent strain functions. Note that strains cannot be made totally independent of each other because they are inter-linked by strain-displacement equations. Ashwell & Sabir[6] have indicated a specific advantage of the elements based on generalised strain functions. Accordingly, if the contribution of the strain energy to the total potential energy is to be minimised (The Principle of Minimum Potential Energy), the variation of strains should be as smooth as possible. Otherwise, since the strain energy is calculated from squares and products of strains, imposing local variations on an initially smooth distribution without altering the mean value will increase the value of squares when they are integrated over the element, thus causing a loss in accuracy when the potential energy is computed.

1.2.1 Application to Circular Arches

The use of strain based finite elements was first introduced by Sabir et al[7] by applying it to a circular arch. Simple functions were assumed for strains and a comparison with classical displacement models showed the superiority of this technique. The experiment was performed on four different types of arches and all types showed a faster rate of convergence.

1.2.2 Application to Cylindrical Shells

Ashwell & Sabir[6] used this method to develop a curved finite element for cylindrical shells. The element derived was rectangular with four external nodes. Each node has five degrees of freedom (three translations and two rotations) giving a total of twenty degrees of freedom per element. This element has been used to solve the pinch cylinder problem and to analyse roof type barrel vaults, and the convergence obtained was superior to that of other elements for which results were available.

1.2.3 Application to Shallow Shells of General Shapes

Savir et al[8] have developed a simple curved finite element based on the strain approach. The element is rectangular and represents exact rigid body, strain free displacements. The degree of convergence obtained here is highly competitive with that obtained in complex higher order elements. However, the element is derived using shallow shell formulation which gives a simplified version of the strain-displacement equations.

1.2.4 Application to Shells of Revolution

In this paper, a Finite Element model containing a curved ring element with two nodal circles having three degrees of freedom per node (figure 1) which is based on assumed strain functions is presented and exact strain free rigid body displacements are introduced in the derivation. The formulation is based on general shell theory[9] and the shallow shell approximation is not made.

2.0 Equations of Elasticity for a Thin Shell of Revolution

The analysis carried out here is limited to *Thin* shells and the theory used is based on the improved first approximation theory of Love, which was presented by Sanders[9]. This technical report by Sanders gives all necessary equations in a generalised way and they are simplified using appropriate paramenters for a shell of revolution.

2.1 Strain-Displacement (Kinematic) equations

The strain displacement equations for the axisymmetric analysis can be obtained by reducing the general set of equations given by Sanders[9] and there are only four strain terms in the equations.

$$\varepsilon_\phi = \frac{1}{R_\phi}\frac{du}{d\phi} + \frac{w}{R_\phi} \quad (1)$$

$$\varepsilon_\theta = \frac{1}{R_\theta}\left[u cot\phi + w\right] \quad (2)$$

$$\kappa_\phi = \frac{1}{R_\phi^2}\left[\frac{du}{d\phi} - \frac{d^2w}{d\phi^2}\right] \quad (3)$$

$$\kappa_\theta = \frac{cot\phi}{R_\phi R_\theta}\left[u - \frac{dw}{d\phi}\right] \quad (4)$$

2.2 Constitutive Equations (Stress-strain relationships)

The mechanical properties of the shell material are incorporated in the stress-strain relationships. Only the equations for an axisymmetric isotropic shell of revolution are given here.

$$N_\phi = \frac{Eh}{(1-\nu^2)}\left[\varepsilon_\phi + \nu\varepsilon_\theta\right] \quad (5)$$

$$N_\theta = \frac{Eh}{(1-\nu^2)}\left[\varepsilon_\theta + \nu\varepsilon_\phi\right] \quad (6)$$

$$M_\phi = \frac{Eh^3}{12(1-\nu^2)}\left[\kappa_\phi + \nu\kappa_\theta\right] \quad (7)$$

$$M_\theta = \frac{Eh^3}{12(1-\nu^2)}\left[\kappa_\theta + \nu\kappa_\phi\right] \quad (8)$$

3. Derivation of the Model

The strains given in 2.1 are expressed in terms of two displacements (*u*,*w*) and hence for consistency, these equations should satisfy two compatibility equations. Compatibility relationships can be obtained by eliminating *u* and *w* from these equations.

$$\frac{d\kappa_\theta}{d\phi} + \frac{\kappa_\theta}{sin\phi cos\phi} - \frac{R_\phi}{R_\theta}cot\phi\kappa_\phi = 0 \quad (9)$$

$$\frac{d\varepsilon_\theta}{d\phi} + cot\phi\,\varepsilon_\theta - \frac{R_\phi}{R_\theta}cot\phi\,\varepsilon_\phi + \frac{R_\phi\kappa_\theta}{cot\phi} = 0 \quad (10)$$

The derivation is basically divided into two parts, namely the rigid body (zero strain) part and the part corresponding to non zero strain modes.

The basic variable considered here is ϕ (meridional angle) and the two principal radii of curvature (R_ϕ, R_θ) are assumed to be constant within an element.

3.1 Rigid Body Displacements

According to Dawe[10] and Cantin[11], the components in a displacement field which represent rigid body modes should give a strain free state of motion and should exhibit all possible rigid body modes of the element.

Equating the strains in equations (1)-(4) to zero, the zero strain state can be obtained.

$$\frac{du}{d\phi} + w = 0 \tag{11}$$

$$u\cot\phi + w = 0 \tag{12}$$

$$\frac{du}{d\phi} - \frac{d^2w}{d\phi^2} = 0 \tag{13}$$

$$u - \frac{dw}{d\phi} = 0 \tag{14}$$

Solving equations (11) & (12);

$$\frac{du}{d\phi} = u\cot\phi$$

$$\int \frac{du}{u} = \int \cot\phi d\phi$$

$$ln(u) = ln(\sin\phi) + C$$

$$= ln(\sin\phi) + ln(\alpha_1)$$

$$= ln(\alpha_1 \sin\phi)$$

$$u = \alpha_1 \sin\phi$$

From equation (11):

$$w = -\alpha_1 \cos\phi$$

Hence terms corresponding to rigid body motion are;

$$u = \alpha_1 \sin\phi \tag{15}$$

$$w = -\alpha_1 \cos\phi \tag{16}$$

It is important to note that u and w in equations (15) & (16) satisfy all zero strain conditions given by equations (11)-(14) thus giving an outright zero strain solution. Also, they give a constant displacement in the axial direction of the shell, which can be expressed as follows (also see Figure 1).

$$axial\ deflection = u\sin\phi - w\cos\phi$$

$$= \alpha_1 \sin^2\phi - (-\alpha_1 \cos^2\phi)$$

$$= \alpha_1\ (constant)$$

This is an explicit representation of the rigid body motion[10]

3.2 Other Strain States

The generalised displacement field should contain 6 independent coefficients in the axisymmetric analysis as there are 6(=3x2) degrees of freedom per element (see figure 1). Since one coefficient α_1 is already allocated, the balance of 5 coefficients have to be apportioned among the four strain quantities. These 5 coefficients are allocated to strains ε_ϕ and k_ϕ since, in axisymmetric analysis, they are the predominant strain terms.

Using a similar approach to that used by Ashwell et al[7] in formulating a strain based finite element model for circular arches, the independent strains are assumed by the following expressions.

$$\varepsilon_\phi = \alpha_3 + \alpha_2\phi \tag{17}$$

$$\kappa_\phi = \alpha_6 + \alpha_5\phi + \alpha_4\phi^2 \tag{18}$$

Eliminating the term $du/d\phi$ from equations (1) & (3);

$$\frac{d^2w}{d\phi^2} + w = R_\phi \varepsilon_\phi - R_\phi^2 \kappa_\phi \qquad (19)$$

The equation (19) is a non homogeneous linear differential equation of order 2,. The solution of this equation has two parts, the *Complementary function* and the *Particular Integral*. The complementary function can be obtained by equating the right hand side of the equation to zero, thus making it a homogeneous equation. The complete solution for displacements can be obtained by superimposing the complementary function and the particular integral[7].

3.2.1 The Complementary function

$$\frac{d^2w}{d\phi^2} + w = 0 \qquad (20)$$

Substituting $w = Ae^{m\phi}$ (A is a constant and m is a factor to be determined), equation (20) can be solved as follows.

$$m^2 + 1 = 0$$

$$m = +i \; and \; -i \quad (i = \sqrt{-1})$$

Hence;

$$w = A_1 e^{i\phi} + A_2 e^{-i\phi}$$

$$= A_1(cos\phi + isin\phi) + A_2(cos\phi - isin\phi)$$

$$= (A_1 + A_2)cos\phi + (A_1 i - A_2 i)sin\phi$$

If we make $A_1 = A_2 = a½$, then

$$w = -\alpha_1 cos\phi \qquad (21)$$

It is interesting to note that this is the same form of expression for w given in equation (16) which corresponds to zero strain (rigid body motion) solution.

3.2.2 Particular Integral

Incorporating the differential operator $D[= d/d\phi]$ in equation (19);

$$(D^2 + 1)w = R_\phi \varepsilon_\phi - R_\phi^2 \kappa_\phi$$

$$w = (1 + D^2)^{-1} \left[R_\phi \varepsilon_\phi - R_\phi^2 \kappa_\phi\right]$$

Using the binomial expansion for negative integers;

$$w = (1 - D^2 + D^4 - .....) \left[R_\phi \varepsilon_\phi - R_\phi^2 \kappa_\phi\right]$$

$$= \left[R_\phi \varepsilon_\phi - R_\phi^2 \kappa_\phi\right] - D^2 \left[R_\phi \varepsilon_\phi - R_\phi^2 \kappa_\phi\right]$$

Substituting expressions for $\varepsilon\phi$ and $K\phi$;

$$w = [R_\phi \phi]\,\alpha_2 + [R_\phi]\,\alpha_3 + \left[(2 - \phi^2)R_\phi^2\right]$$

$$\alpha_4 + \left[-R_\phi^2 \phi\right]\alpha_5 + \left[-R_\phi^2\right]\alpha_6 \qquad (22)$$

Using the equation (1) the corresponding expression for u can be obtained.

$$u = [R_\phi]\,\alpha_2 + [0]\,\alpha_3 + \left[-R_\phi^2(2\phi - \frac{\phi^3}{3})\right]$$

$$\alpha_4 + \left[R_\phi^2 \frac{\phi^2}{2} - R_\phi^2\right]\alpha_5$$

$$+ \left[R_\phi^2 \phi\right]\alpha_6 \qquad (23)$$

The complete solution is obtained as given below by combining the complementary function and the particular integral.

$$u = [sin\phi]\,\alpha_1 + [R_\phi]\,\alpha_2 + [0]\,\alpha_3 + \left[-R_\phi^2(2\phi - \frac{\phi^3}{3})\right]\alpha_4$$
$$+ \left[R_\phi^2 \frac{\phi^2}{2} - R_\phi^2\right]\alpha_5 + \left[R_\phi^2 \phi\right]\alpha_6 \qquad (24)$$

$$w = [-cos\phi]\,\alpha_1 + [R_\phi \phi]\,\alpha_2 + [R_\phi]\,\alpha_3 + \left[(2 - \phi^2)R_\phi^2\right]\alpha_4$$
$$+ \left[-R_\phi^2 \phi\right]\alpha_5 + \left[-R_\phi^2\right]\alpha_6 \qquad (25)$$

The equations (24) & (25) give the displacement functions to be used in the finite element analysis. In this derivation, strain functions are smooth polynomials and displacement functions are based on these assumed strain functions. The remaining strain terms ε_θ and k_θ can be obtained by substituting above displacement fields in strain-displacement equations given by (2) & (4).

4.0 Finite Element Analysis

Once the displacement functions are known, the analysis is carried out using the standard finite element analysis procedure. The description of the procedure is not discussed here and any text book on Finite Element Method may be referred for details.

5.0 Geometry of Shell Element

By reducing the expression for geometry used by Stricklin et al[4], ϕ can be expressed as a linear function of s, where s is the non dimensional length along the meridian which is used in numerical integration.

$$\phi = C_0 + C_1 s \qquad (26)$$

$$\begin{aligned} &where \\ &C_0 = \phi_1 \\ &C_1 = (\phi_2 - \phi_1) \end{aligned}$$

(ϕ_1 and ϕ_2 are values of ϕ at nodal circles).

6.0 Numerical Examples

Two numerical examples are given here to illustrate the performance of the shell element based on assumed strain functions. The analyses done with the element derived here are compared with analyses done with normal ring shell elements based on classical derivation (assumed displacement functions) given in the references cited.

1. Clamped hemispherical shell (open crowned) with an applied tip moment.

The geometry of this shell and other details required in the analysis are given in figure 2. Figures 4 and 5 give the comparison of results for horizontal displacement ($\bar{w}$) and meridional moment M_ϕ with reference to results of Grafton & Strome[2].

2. Hyperbolic shell (Ardeer cooling tower) under dead load.

The details of this shell are given in figure 3. Figures 6 and 7 give the comparison of results for normal displacement (w) and circumferential stress resultant N_θ with respect to results of Lam[12]. It should be noted that higher order elements are used with a displacement function approach in Lam's derivation.

7.0 Conclusions & Recommendations for future developments

The present finite element model which is derived using assumed strain functions is performing satisfactorily as demonstrated by the numerical examples considered. The formulation is done for shells which are loaded axisymmetrically and this model cannot take into account non-symmetrical loads acting on the shell. This reduces the spectrum of applications of this ring element, but nevertheless the element has been successfully developed. This initial derivation can serve as a basis for future developments.

The element can be further developed to incorporate asymmetrical loads (wind loads) which will enhance the usage of the element. Finally the true potential of this stain based approach can be investigated if applied to non linear problems.

8.0 References

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Appendix - List of symbols

E	-	Youngs Modulus
h	-	thickness of element
l	-	curvilinear element length
M_ϕ	-	meridional moment resultant
M_φ	-	circumferential moment resultant
N_ϕ	-	meridional stress resultant
N_φ	-	circumferential stress resultant
R_ϕ	-	meridional radius of curvature
R_φ	-	circumferential radius of curvature
r	-	horizontal radius
S	-	meridional distance along shell surface
s	-	dimensionless meridional distance
u	-	*meridional displacement*
v	-	circumferential displacement
w	-	normal displacement
x	-	axial distance
β_ϕ	-	meridional rotation
ε_ϕ	-	meridional inplane strain
ε_φ	-	circumferential inplane strain
ϕ	-	meridional angle
θ	-	circumferential angle
ν	-	Poissons ratio
k	-	meridional bending strain
k	-	circumferential bending strain

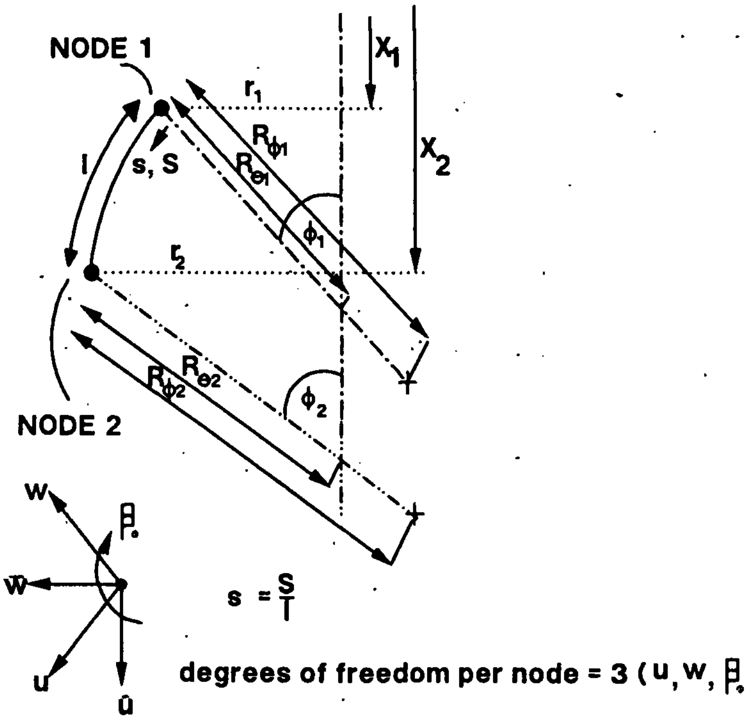

FIGURE 1 - Nodal Geometric parameters

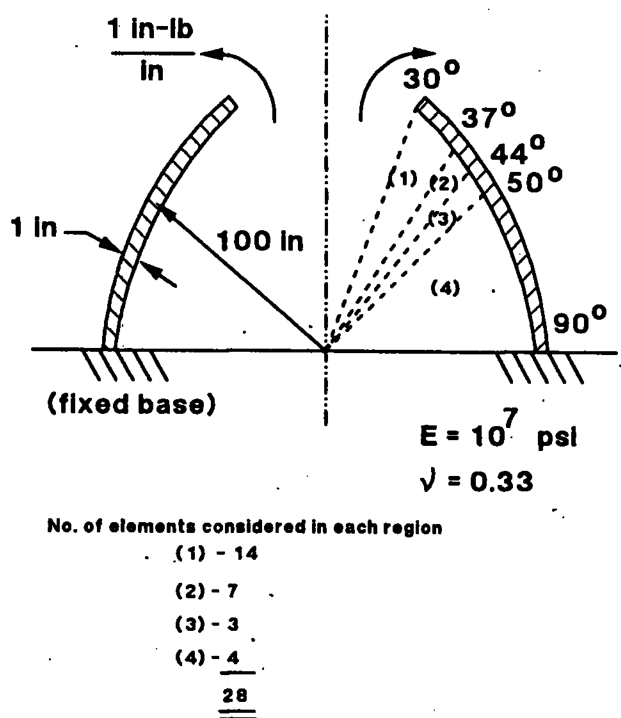

FIGURE 2 - Hemispherical Shell with tip moment

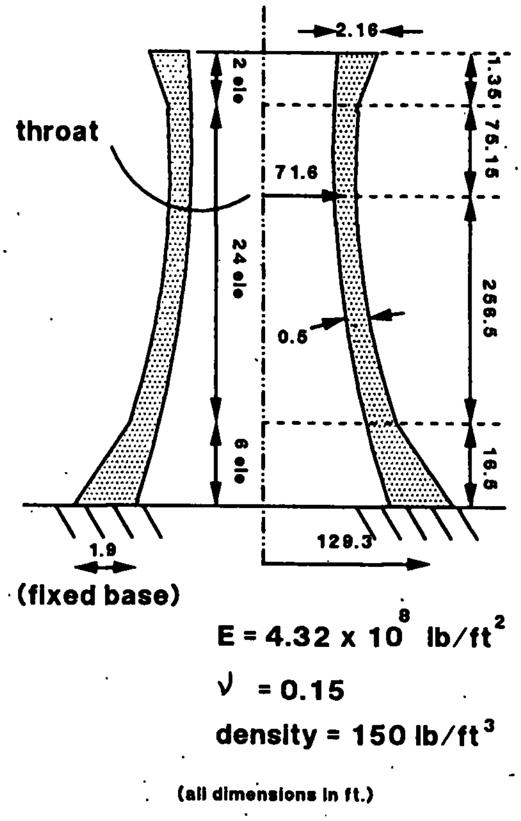

FIGURE 3 - Ardeer Cooling Tower (hyperbolic shell)

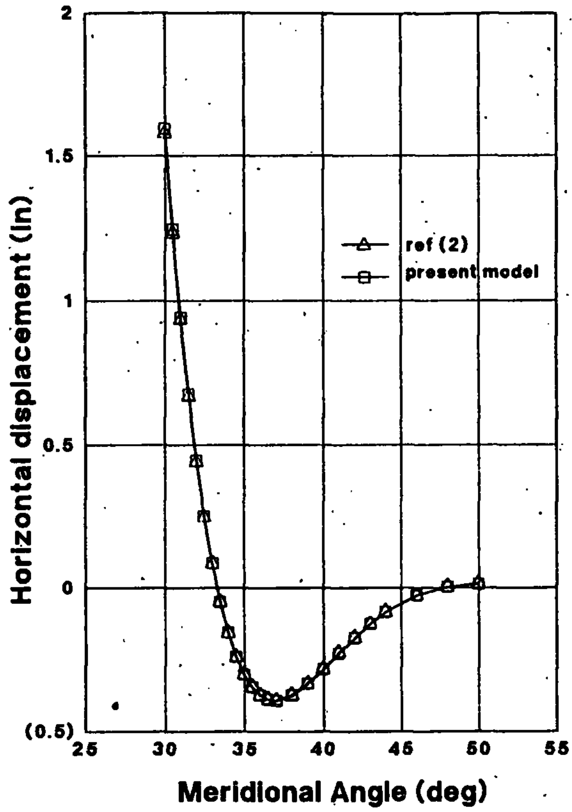

FIGURE 4 - Comparison of Horizontal diplacement

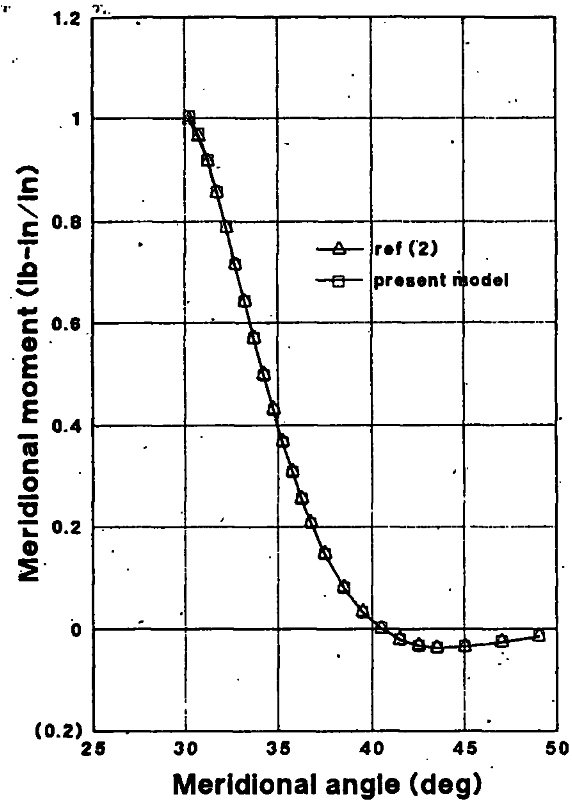

FIGURE 5 - Comparison of Meridional moment

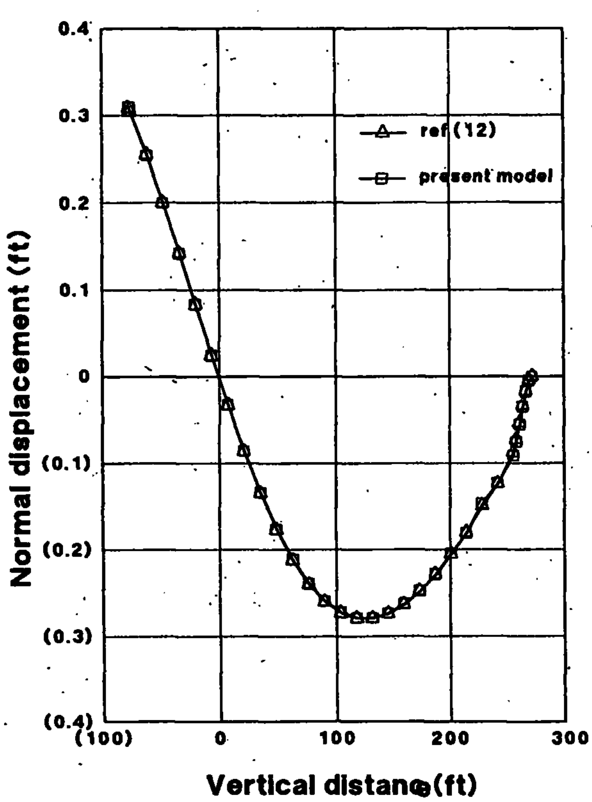

FIGURE 6 - Comparison of Normal displacement

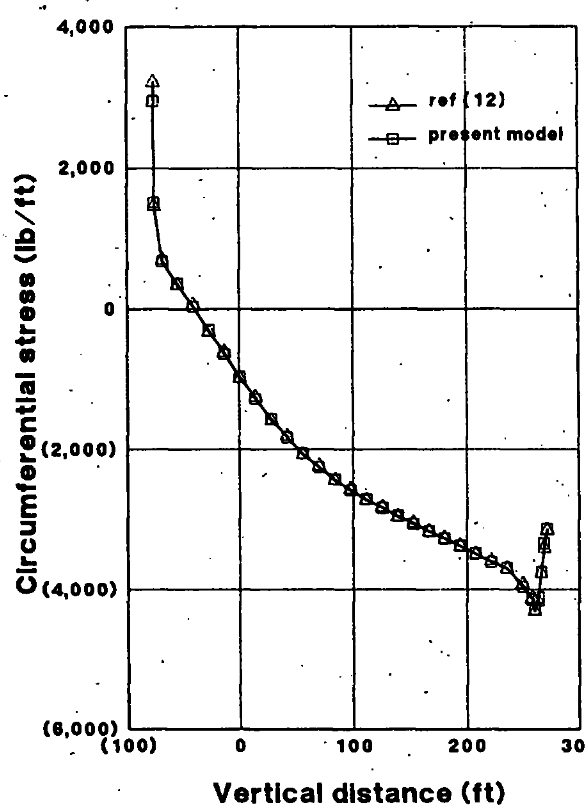

FIGURE 7 - Comparison of Circumferential stress