DYNAMICS OF DISTENSIBLE WAVEGUIDES

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NOMENCLATURE

$C_o, C(s), C'$ = phase speed
$D = 2a$ = diameter of waveguide
E, k = Young's modulus, bulk modulus
$f(t)$ = forcing function
G = influence function
h = wall thickness
J, I = Bessel functions
$M_o, M(s), M'$ = Mach number
p = pressure
R = Reynolds number
z, r = cylindrical coordinates
s = Laplace variable
t = time
U = reference velocity
u_r, u_z = velocity components
V = average velocity
$\Gamma(\ )$ = impedence function
$\delta(\ )$ = Dirac delta function
ϵ, η = deflections of wall
η, ξ = dummy variables
ν = kinematic viscosity
ρ_t, ρ = density of wall and fluid
σ = Poissons ratio
ϕ = elasticity number
ω, ω_n = frequency of excitation, natural frequency

Introduction

The frequency response of distensible waveguides can be considered to be completely solved in relation to several problems in biomedical engineering.[1] In contrast, however, the response of distensible viscous fluid lines to isolated disturbances has received little attention although its relevance to biomechanics and fluidic systems is significant. The main reason for this lapse can be attributed to the inherent complexities of the governing equations and the absence of a suitable analytical technique for their simultaneous synthesis.

Viscous decay of fluid transients was first treated analytically by Wood,[2] in 1937, with the solution of the one-dimensional telegrapher's equation using integral transformations. In this, he used a constant dissipative coefficient based on Poiseuille flow to account for fluid friction, thereby implying that flow is fully developed in transient situations. This assumption, however, was in contradiction to the experimental demonstration of Richardson and Tyler.[3] In order to overcome this shortcoming of the one-dimensional model, Iberall,[4] in 1950, proposed an improved model based on Kirchoff's equations of sound.[5] The method effectively accounted for non-uniform effects of flow establishment, although without actually predicting them, and thus dispensed with the concept of a constant, frequency-independent friction coefficient. Albeit Iberall's work formed the basis for much of the work that followed on the transient response of fluid lines [6,7,8] analyses were confined to elastic but non-dynamic waveguides. In situations such as bloodflow, however, the distensible nature of the arteries made it mandatory to consider its dynamics and the frequency response of a model of the vascular system was solved by Womersley,[5] in 1957. Thus, while Iberall's work led to the transient response of non-distensible waveguides, Womersley provided the frequency response for a distensible waveguide.

The following analysis, which fills the gap between the solutions of Iberall's model and Womersley's solution, is based on a hybrid between their mathematical models. It yields closed-form solutions for the transient response of distensible, viscous fluid lines which gave good correlation with experimental results.

The particular problem studied is that of signal transmission in a distensible, semi-infinite waveguide of circular cross-section for the physical situation depicted in Fig. 1, viz., a piston conveys an arbitrary velocity impulse to a compressible, non-heat-conducting Newtonian fluid and it is desired to obtain the resulting velocity and pressure responses in the fluid for all time and space.

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2.0 Mathematical Set

The Iberall model for transient fluid flow can be obtained by suitably modifying the Oseen equations[9]. The necessary simplifications are based on the assumptions [9,10] that—

(a) the fluid is initially at rest,
(b) the flow is axi-symmetrical,
(c) the pressure is constant across the cross-section,
(d) wave dispersion due to volumetric dilatation is negligible, and
(e) the fluid is barotropic, i.e. $\rho = \rho(p)$ and, hence,

$$\frac{\partial p}{\partial \rho} = \frac{K}{\rho_o} = C_o^2 \qquad \text{.....................(1)}$$

where, k = bulk modulus of elasticity, p = pressure, ρ = density, and subscript o denotes reference conditions.

Defining u_z = velocity in z-direction, u_r — velocity in r-direction, D = diameter, and U = a suitable characteristic velocity, with respect to Fig. 1, the following non-dimensional variables will be used with advantage:

$$u_r^* = \frac{u_r}{U}; \quad u_z^* = \frac{u_z}{U}; \quad z^* = \frac{z}{D}; \quad r^* = \frac{r}{D};$$
$$p^* = \frac{p}{\rho U^2}; \quad t^* = \frac{t}{D/U}; \quad \rho^* = \frac{\rho}{\rho_o} \qquad \text{.....................(2)}$$

Utilizing the assumption of Eq. (1) and the variables of (2), and omitting the superscript asterisk, the Iberall model can now be written in the form[9]

$$\frac{\partial u_z}{\partial t} + \frac{\partial p}{\partial z} = \frac{1}{R}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right] \qquad \text{...(3)}$$

$$\frac{\partial p}{\partial r} = 0 \qquad \text{...(4)}$$

$$\frac{\partial p}{\partial t} + \frac{1}{M_o^2}\left[\frac{\partial u_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\,u_r)\right] = 0 \qquad \text{...(5)}$$

where, R = UD/ν = characteristic Reynolds number, ν = kinematic viscosity, and $M_o = U/c_o$ = Mach number. Further to the set formed by (3), (4) and (5), the extension to the Iberall model to allow for wall distensibility requires the introduction of a dynamic equation for the wall. A suitable expression written in non-dimensional form is[9]

$$\frac{\partial^2 \eta}{\partial t^2} + \omega_n^2 \eta = \left(\frac{D}{h}\right)\left(\frac{\rho_o}{\rho_t}\right) p\left(r=\tfrac{1}{2}\right) \qquad \text{.........(6)}$$

where, cf. Fig. 1, η = radial displacement of waveguide wall, w_n — radial natural frequency of waveguide, ρ_t = density of waveguide material, and h = wall thickness. Equation (6) is based on the assumptions that

$$\sigma\frac{\partial \eta}{\partial z} \ll \frac{\partial^2 \epsilon}{\partial z^2}; \quad \sigma\frac{\partial \epsilon}{\partial z} \ll \eta; \quad \frac{h}{D} \ll 1 \qquad (7)$$

where, σ = Poisson's ratio and ϵ = axial displacement (cf. Fig. 1).

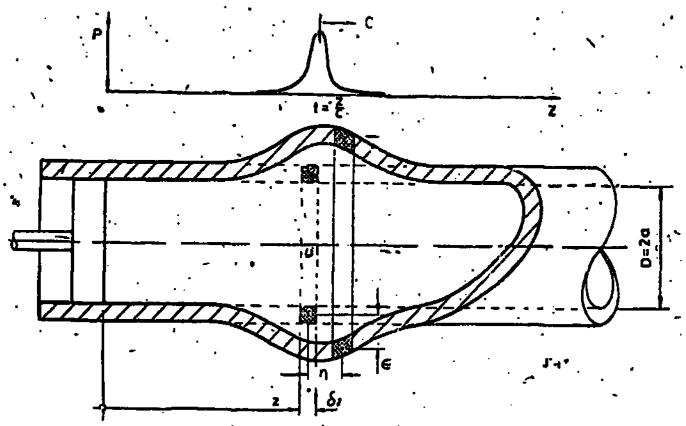

Fig. 1

To complete the set, it remains to state the boundary and initial conditions. These are:

(i) The particle velocity at the wall in the axial direction is zero, i.e.,

$$U_z(z, \tfrac{1}{2}, t) = 0; \quad z > 0 \qquad \text{..............(8)}$$

(ii) The average velocity of fluid particles

$$V(z,t) = 8\int_0^{\frac{1}{2}} u_z(z,r,t)\,r\,dr \qquad \text{...............(9)}$$

at z = 0 will be an arbitrary function of time, i.e.,

$$V(0,t) = f(t) \qquad \text{................(10)}$$

(iii) The initial velocity and pressure fields are stationery, viz.

$$u_z(z,r,0) = \frac{\partial u_z}{\partial t}(z,r,0) = 0 \qquad \text{..........(11)}$$

and

$$p(z,0) = 0 \qquad \text{..............(12)}$$

(iv) The radial fluid velocity at the wall is equal to the radial wall velocity, i.e.,

$$u_r\left(r=\tfrac{1}{2}\right) = \frac{\partial \eta}{\partial t} \qquad \text{...............(13)}$$

(v) The waveguide wall is initially at rest, i.e.,

$$\eta(t=0) = \frac{\partial \eta}{\partial t}(t=0) = 0 \qquad \ldots\ldots\ldots\ldots\ldots(14)$$

Equations (3) through (6), (8) and (10) through (14) comprise the complete mathematical set to be integrated.

3.0 The Associated Problem

The integration of the mathematical set can be accomplished by first solving the associated Green's function problem in which the inhomogeneous boundary condition (10) is replaced with a Dirac delta-function at $t = \xi$[12], i.e.,

$$G_u(o,r,t;\xi) = \delta(t-\xi) \qquad \ldots\ldots\ldots\ldots\ldots(15)$$

where, G_u = influence function for the axial velocity. The influence functions for the radial velocity and pressure are defined as G_V and G_p, respectively.

Using the technique developed by Iberall[4,9], a solution of (3) in the Laplace domain can be obtained in the form

$$\bar{G}_u(z,r,s;\xi) = A(z,s;\xi)\, I_0(\sqrt{sR}\, r) - \frac{1}{s}\frac{\partial \bar{G}_p}{\partial z}(z,s;\xi) \qquad \ldots\ldots\ldots\ldots\ldots(16)$$

where, A = constant of integration, I_0 = modified Bessel function of the first kind and order zero, s = Laplace variable, and overbar denotes a transformed dependent variable. By applying the no-slip condition at the wall Eq. (8), (16) reduces to

$$\bar{G}_u(z,r,s;\xi) = A(z,s;\xi)\left[I_0(\sqrt{sR}\, r) - I_0(\sqrt{sR}\, {}^1\!/_2)\right] \qquad \ldots\ldots\ldots\ldots\ldots(17)$$

where the pressure gradient is found to be

$$\frac{1}{s}\frac{\partial \bar{G}_p}{\partial z} = A(z,s;\xi)\, I_0(\sqrt{sR}\, {}^1\!/_2) \qquad \ldots\ldots\ldots\ldots(18)$$

The radial dependency of (17) can now be eliminated by averaging $\bar{G}_u$[9] according to (9) yielding

$$\bar{G}_v(z,s;\xi) = -A(z,s;\xi)\, I_2(\sqrt{sR}\, {}^1\!/_2) \qquad \ldots\ldots\ldots\ldots(19)$$

where, $\bar{G}_V$ = associated influence function for the average axial velocity, V, and I_2 = modified Bessel function of the first kind and order two.

The second stage of the integration constitutes the manipulation of the continuity equation which provides a mechanism for introducing wall inertia. Eq. (5) in terms of the respective influence functions when Laplace transformed subject to the initial conditions of (11) and (12), and averaged across the cross-section results in

$$s\bar{G}_p M_0^2 + \frac{\partial \bar{G}_v}{\partial z} + 4\,\bar{G}_v(r = {}^1\!/_2) = 0 \qquad \ldots(20)$$

Unlike in the Iberall solution[4], the radial velocity at the wall (third term of Eq. (20)) is non-zero by virtue of wall distensibility and is related to the wall velocity through Eqs. (6) and (13). Subject to (13) and (14), Eq. (6) has the Laplace domain solution[9]

$$\bar{U}_r(r = \tfrac{1}{2}) = \left(\frac{D}{h}\right)\left(\frac{\rho_0}{\rho_s}\right)\left(\frac{-s\bar{p}}{s^2 + \omega_n^2}\right) \qquad \ldots\ldots\ldots\ldots(21)$$

Hence, recognizing the fact that $\bar{u}_r = \bar{G}_V$ and $\bar{p} = \bar{G}_p$, (20) and (21) combine to give

$$s\bar{G}_p M^2(s) + \frac{\partial \bar{G}_v}{\partial z} = 0 \qquad \ldots(22)$$

where,

$$M^2(s) = M_0^2 + 4\left(\frac{D}{h}\right)\left(\frac{\rho_0}{\rho_s}\right)\frac{1}{s^2 + \omega_n^2} \qquad \ldots\ldots\ldots\ldots\ldots(23)$$

defines a Mach number based on the frequency dependent phase speed, c(s), cf. Appendix B. Equation (23) along with Eqs. (17) and (19) subject to initial condition (10) gives the required solution to the associated problem in the Laplace domain, viz.

$$\bar{G}_p = \frac{1}{M_0}\frac{I_0(\sqrt{sR}\, {}^1\!/_2)}{\Gamma(s)\, I_2(\sqrt{sR}\, {}^1\!/_2)}\, e^{-s\xi}\, e^{-sM_0\Gamma(s)z} \qquad \ldots\ldots(24)$$

where,

$$\Gamma(s) = \frac{M(s)}{M_0}\left[\frac{I_0(\sqrt{sR}\, {}^1\!/_2)}{I_2(\sqrt{sR}\, {}^1\!/_2)}\right] \qquad \ldots\ldots(25)$$

is the impedence function of the system (cf. Appendix B).

4.0 Time Domain Solutions

In problems of this nature, solutions in the complex plane are usually possible but exact inversions of these in terms of real-time often pose insurmountable difficulties. Therefore, more often than not, approximate inversions have to be resorted to.

In the case of Eq. (24), the appropriate mode of inversion is not obvious and, in order to obtain an insight to the problem, the inviscid problem will be first considered. Thus, for $R \to \infty$, Eq. (24) reduces to

$$\bar{G}_p = \frac{1}{M_0 \Gamma_i(s)}\, e^{-s\xi}\, e^{-sM_0\Gamma_i(s)z} \qquad \ldots\ldots(26)$$

where,

$$\Gamma_i(s) = \frac{M(s)}{M_0} = \frac{c_0}{c(s)} \qquad \ldots\ldots(27)$$

Although an exact inversion of (26) exists[9], its complexity does not permit any useful inferences and, therefore, approximate inversions for small time and large time, i.e., large s and small s, respectively, will be made. For small time, $\beta_1(s) \rightarrow 1$, whence, the inversion yields the trivial Joukowsky solution[9]

Joukowsky solution[9]

$$G_p = \frac{1}{M_o}\delta(t - M_o z - \xi);\ M_o = \frac{U}{c_o}$$

Similarly, for large time, $\Gamma_1(s) \rightarrow (1+\phi)^{\frac{1}{2}}$, where $\phi = \left(\frac{D}{h}\right)\left(\frac{k}{E}\right)(1-\sigma^2)$ = elasticity number and

$$G_p = \frac{1}{M'}\delta(t - M'z - \xi);\ M' = \frac{U}{c'}$$

where, $M' = M(1+\phi)^{\frac{1}{2}}$ is the Mach number based on the Korteweg phase speed c' (cf. Appendix B). This again is the Joukowsky solution for wave propagation in elastic but non-distensible waveguides. Therefore, wall inertia clearly plays a role at intermediate time. By means of several trial approximations it became evident that it was essentially the presence of $\Gamma(s)$ in the denominator of (26) which accounts for wall inertial effects. Accordingly, expression (26) is simplified somewhat arbitrarily by making small-time approximations of only the second exponential argument and the numerator of the coefficient factor while leaving the denominator unchanged. On this basis, (26) transforms into

$$\bar{G}_p = \frac{1}{M'}\frac{\omega_n(1+\phi)^{1/2}}{[s^2+(1+\phi)\omega_n^2]^{1/2}}\, e^{-s\xi} e^{-sM'z} \qquad \text{.............(30)}$$

For illustrative purposes, for a Heaviside step-input, Eq. (30) has the time-domain solution

$$p(z,t) = \frac{\omega_n(1+\phi)^{1/2}}{M'}\int_{M'z}^{t} J_o[(1+\phi)^{1/2}\omega_n(\eta - M'z)]\,d\eta \qquad (31)$$

where J_o = Bessel function of the first kind and order zero and η is a dummy variable of integration. As would be expected from an underdamped system the solution (31) is oscillatory (curve 2 of Fig. 2). Moreover the oscillations take place about the classical Joukowsky solution for an inviscid non-inertial system (curve 1). It is of interest to note that the pressure-maximum periodically exceeds the generally accepted maximum given by the Joukowsky pressure.

The above procedure offers a basis for the approximate inversion of (24). The Bessel$_o$ functions in $\Gamma_2 = \Gamma/\Gamma_1$ are expanded asymptotically for large s, i.e.,

$$\lim_{s\to\infty} \Gamma_2(\sqrt{sR}/4) \rightarrow \left[1 + \frac{4}{Rs} + \left(\frac{4}{Rs}\right)^{3/2} + \frac{7}{8}\left(\frac{4}{Rs}\right)^{5/2} + \cdots\right] \qquad \text{...(32)}$$

With expanded as in (30), a tractable form of (24) can now be obtained, viz.

$$\bar{G}_p(z,s;\xi) = \frac{1}{M'}\frac{\omega_n(1+\phi)^{1/2}}{[s^2+(1+\phi)\omega_n^2]^{1/2}}\, e^{-s\xi} e^{-sM'z} e^{-M'z\left(\frac{4s}{R}\right)^{1/2}} e^{-\frac{4M'z}{R}} \qquad \text{...(33)}$$

which has the inversion[9]

$$G_p(z,t;\xi) = (1+\phi)\omega_n^2\, e^{-\frac{M'z}{R}} \int_{M'z+\xi}^{t} J_1[(1+\phi)^{1/2}\omega_n(t-\eta)] \cdots \qquad \text{...(34)}$$

$$\cdots \operatorname{Erfc}\left[\left(\frac{M'z}{R}\right)^{1/2}\left(\frac{M'z}{\eta - M'z - \xi}\right)^{1/2}\right] d\eta$$

where, J = Bessel function of the first kind and order one, and Erfc = complementry error function. Using the convolution[18]

$$p(z,t) = \int_0^t G(z,t;\xi) f(\xi)\, d\xi \qquad \text{...(35)}$$

the integral representation of the pressure response of a distensible viscous fluid line due to an arbitrary velocity input, as described by (10), is obtained as

$$p(z,t) = \frac{(1+\phi)^{1/2}\omega_n}{M'} e^{-\frac{4M'z}{R}} \int_0^t \int_{M'z}^{\xi} J_o[(1+\phi)^{1/2}\omega_n(t-\eta)] \cdots$$

$$\cdots \operatorname{Erfc}\left[\left(\frac{M'z}{R}\right)^{1/2}\left(\frac{M'z}{\eta - M'z}\right)^{1/2}\right] d\eta\, f'(t-\xi)\, d\xi \qquad \text{...............(36)}$$

In (36), which is the required time-domain solution, f'() refers to the derivative of f() with respect to its argument. For comparison purposes, the equivalent solution for the elastic but non-distensible case, i.e. the standard Iberall model solution, is given by[9]

$$p(z,t) = \frac{e^{-\frac{4M'z}{R}}}{M'} \int_{M'z}^{t} \operatorname{Erfc}\left[\left(\frac{M'z}{R}\right)\left(\frac{M'z}{\eta - M'z}\right)\right] f(t-\eta)\, d\eta \qquad \text{...............(37)}$$

To reveal the characteristics of these solutions, their responses are included in Fig. 2 where, as would be expected, it is seen that the extended Iberall model solution (curve 4) for a Heaviside step-input oscillates with decreasing amplitude about the corresponding Iberall model solution (curve 3).

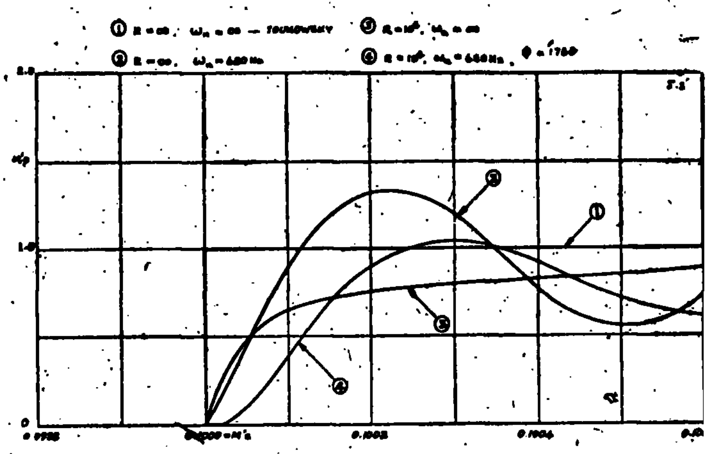

Fig. 2

5.0 Discussion

The normalized results of a limited experimental investigation of the decay of the maximum value of an arbitrary pulse propagating in a distensible waveguide[9] are plotted in Fig. 3. Also included are the predictions based on Eqs. (36) and (37) for a parabolic disturbance with the elastic values of the tube the same. Although the points are in satisfactory agreement, it was found that the two curves coincided due to the fact that the duration of the pulse was comparatively larger than the natural frequency of the waveguide ($1/\omega_n$). To be sure, for cases when $1/\omega_n$ is larger than the pulse duration, opposite would be the results.

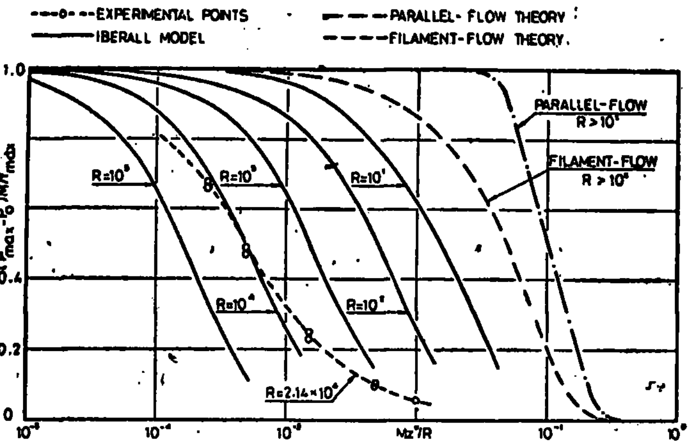

Fig. 3

From the predictions based on the one-dimensional telegrapher's equation-[18] and the author's parallel-flow model 14.15 included in Fig. 3, is quite clear that they give grossly erroneous results (cf. Appendix A).

A clue to the success of solutions based on Iberall's model can be found in the works of Gerlach 11.16 where he demonstrated that the Iberall model, in fact, gives solutions corresponding to the first mode of the eigenvalue solution which is partially obtainable from attempts to solve the Oseen equations for waveguide problems. The profile of the first mode must be, therefore, closer to the actual profile than a parabola and, furthermore, it is highly frequency dependent. This fact was verified indirectly by the author by devising a frequency-dependent friction coefficient for the one-dimensional telegrapher's equation based on the Iberall model which was found to be quite satisfactory for naturally periodic flows-9.17.

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APPENDIX A

The Parallel-Flow Model

All analyses on wave propagation have neglected one aspect, viz. the non-uniform effects of boundary layer development behind the wave front. In order to obtain a mathematical description of this phenomena it is necessary to retain the three-dimensional nature of the model, throughout its analysis. Towards this end, using the same non-dimensional variables of (2), the Navier-Equations can be series-expanded in terms of the small parameter M/R. This results in [14,16,18]

$$\frac{\partial u_z}{\partial t} + \frac{\partial p}{\partial z} = \frac{1}{R}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right] \qquad \text{.............(38)}$$

$$\frac{\partial p}{\partial r} = 0 \qquad \text{.............(39)}$$

$$\frac{\partial p}{\partial t} + \frac{1}{M^2}\frac{\partial u_z}{\partial z} = 0 \qquad \text{.............(40)}$$

Using Laplace transforms and separation of variables, (38), (39) and (40) can be integrated subject to the boundary and initial conditions described in the paper to yield the eigenvalue solution[15]

$$u_z(z,r,t) = 2\sum_{n=1}^{\infty}\frac{J_{o,n}(2j_{o,n}r)}{j_{o,n}J_1(j_{o,n})}\, e^{-2j_{o,n}^2\frac{M'z}{R}} \cdots$$

$$\cdots\left[\frac{2j_{o,n}^2 Mz}{R}\int_{M'z}^{t} e^{-\frac{2j_{o,n}^2\xi}{R}}\, I_1\left[\frac{2j_{o,n}^2}{R}\sqrt{\xi^2-M'^2z^2}\right]\frac{f(t-\xi)\,d\xi}{[\xi^2-M'^2z^2]} + e^{-2j_{o,n}\frac{Mz}{R}} f(t-Mz)\right] \qquad \text{.............(41)}$$

$$p(z,r,t) = \frac{2}{M}\sum_{n=1}^{\infty}\frac{J_{o,n}(2j_{o,n}r)}{j_{o,n}J_1(j_{o,n})}\int_{Mz}^{t} e^{-\frac{2j_{o,n}^2\xi}{R}}\, I_o\left[\frac{2j_{o,n}^2}{R}\sqrt{\xi^2-M^2z^2}\right]$$

$$\cdots\left[4\frac{j_{o,n}^2}{R}f(t-\xi) - \frac{d}{d\xi}f(t-\xi)\right]d\xi \qquad \text{.............(42)}$$

where, $j_{o,n}$ is the nth zero of the zero-order Bessel function of the first kind, and $J_{o,n}$ is the associated nth eigenfunction.

Although as far as pulse decay is concerned the predictions of (42) were found to be unsatisfactory (cf. Fig. 3). Eq. (40) gave a lucid qualitative description of transient flow establishment as seen from Fig. (4). The shortcoming of this model lies in the not so obvious violation of continuity where the radial velocity gradients of Eq. (5) get eliminated in deriving (40).

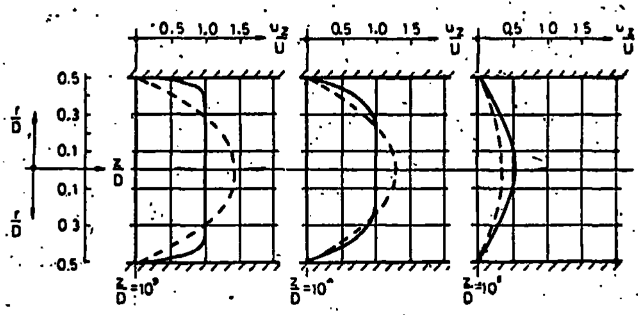

Fig. 4

APPENDIX B

Frequency Dependent Phase Speed

The impedance function $\Gamma(s)$ contains all the dynamic properties which governs the response of the system.[9] Hence, in the rearrangement of Eq. (25), i.e.,

$$\Gamma(s) = \Gamma_1(s).\Gamma_2(s) = \left[\frac{c_o}{c(s)}\right]\left[\frac{I_o(\sqrt{s}R/2)}{I_2(\sqrt{s}R/2)}\right]^{1/2}$$

$$= \left[\frac{1+\frac{s^2}{\omega_n^2}}{1+\frac{s^2}{\omega_n^2}+\left(\frac{D}{h}\right)\left(\frac{k}{E}\right)(1-\sigma^2)}\right]^{1/2}\left[\frac{I_o(\sqrt{s}R/2)}{I_2(\sqrt{s}R/2)}\right]^{1/2} \qquad \text{.............(43)}$$

$\Gamma_1(s)$ represents frequency effects on the phase speed while $\Gamma_2(s)$ describes frequency dependent viscous dispersion and decay. Rewriting $\Gamma_1(s)$ in the form

$$\frac{1}{\Gamma_1(\omega)} = \frac{c(\omega)}{c_o} = \left[\frac{1+s^2/\omega_n^2}{1+s^2/\omega_n^2+(D/h)(k/E)(1-\sigma^2)}\right]^{1/2}$$

it becomes obvious that when $\omega_n \rightarrow \infty$ for non-distensible tubes,

$$\frac{1}{\Gamma_1(\omega)} \rightarrow \frac{1}{[1+(\frac{D}{h})(\frac{k}{E})(1-\sigma^2)]^{1/2}} = \frac{c'}{c_o} \qquad \text{.............(44)}$$

which is the Korteweg equation for phase speed for elastic but not non-distensible tubes. For distensible waveguides, however, it is seen that frequency effects cannot be ignored and this aspect has had previous attention in references 1 and 20.

The transient behaviour in the Laplace domain can be converted to the frequency-domain by the simple transformation

$$s = i\omega$$

where $i = \sqrt{-1}$. Hence, from (44),

$$c(\omega) = c_o\left[\frac{1-\omega^2/\omega_n^2}{1+(D/h)(k/E)(1-\sigma^2)-\omega^2/\omega_n^2}\right]^{1/2} \qquad \text{.............(45)}$$

It follows from this that all frequencies which make c(w) complex will get suppressed since there are no real phase speeds corresponding to them and, hence, a distensible waveguide will function as a rejection acoustic filter. The frequency-dependent behaviour of the phase speed is graphically portrayed in Fig. (5) and this phenomenon can be directly transposed to the transient time-domain if one remembers that steep velocity gradients in the forcing function correspond to high frequency Fourier components. Therefore, for smoothly varying disturbances where, $w << w_n$ the inertia effects of the waveguide will be negligible.

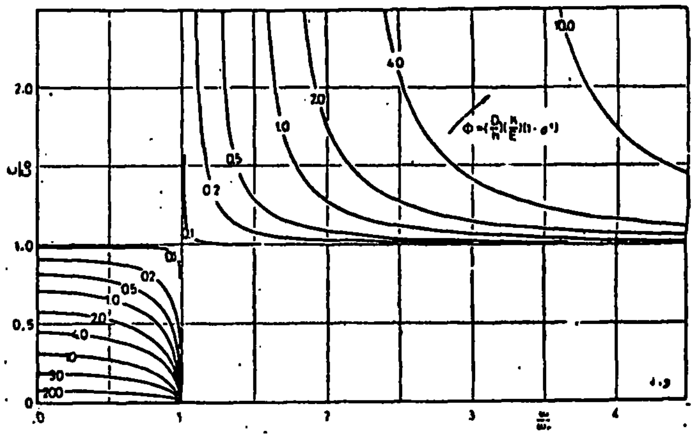

Fig. 5

(*Library—Contd. From Page* 17)

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