ANALYTICAL SOLUTION FOR WATER SURFACE PROFILE IN AN IRRIGATION CANAL UNDER STEADY FLOW CONDITION

by

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Abstract

The practical objective of the analytical solution obtained for water surface profile as a function of depth and distance under steady flow condition for open channel flow in irrigation canal is to develop a simple and comprehensive management tool for operation of irrigation systems. Also one of the theoretical objectives of the analytical solution is to remove the singularity that exists in surface gradients under critical flow conditions where the solution becomes infinity. Furthermore, more accurate boundary conditions for critical flow conditions can be obtained by this method which could be used in numerical methods as well. One of the interesting phenomena observed in the analytical analysis is the gradual rise in water surface profiles in the main canal near the offtakes due to momentum change when water is taken away by the offtake.

1.0 Introduction

New development of irrigation schemes is becoming lesser and less and the attention towards efficient management of existing schemes to obtain the optimum usage of water has become the main objective throughout the world. Irrigation management can be broadly classified in to two categories namely "Main system management" and "Farm level management". According to some researchers most of the fundamental problems are due to poor performance of the main system as the farm level management is mainly dependent on the performance of the main system. One of the major constraints that exists in main system management is the lack of proper management tools for better operational standards. The main system consists of main and branch canals in which the main canal takes off from the main water source. Control structures and offtakes structures are located along the main and branch canals for serving the irrigable area.

The operation of the main system deals with supplying the irrigation water at offtakes with minimum operational losses and setting structures to match with the demand for a given inflow from the source under steady or unsteady flow condition in canal. The methodology adopted in this situation is to solve De Saint- Venants equations of continuity and momentum for given boundary conditions in the canal reach to get the water surface profiles and thus computing offtake openings to achieve targeted demands. The behaviour of the water surface profile in an irrigation canal under steady flow conditions with and without offtake outflow is analyzed in this paper by an analytical approach and the solutions are compared with numerical solutions obtained by 4th order Runge-Kutta method. The governing equations for open channel flow by De Saint-Venant for continuity and momentum are:

$$\frac{\partial A}{\partial t} + \frac{\partial (UA)}{\partial X} = q$$

$$\frac{\partial (UA)}{\partial t} + \frac{\partial (U^2 A)}{\partial X} + gA\frac{\partial h}{\partial X} = gA(S_x - S_f) + qU_q$$

where,

U = Mean cross sectional velocity

A = Cross sectional area of the channel

h = Depth of water

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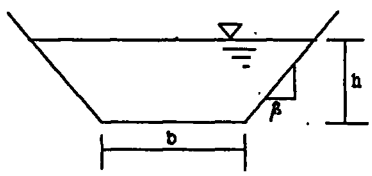
q = Lateral flow (Inflow positive, Outflow negative)
= Downstream component of velocity of the lateral flow
X = Distance
t = Time
Sx = Channel slope
S_f = Friction slope
g = Acceleration of gravity

2.0 Analysis 1: Analytical solution for water surface profile without offtake outflow

Under steady state conditions the continuity and momentum equations for open channel flow without lateral outflow from an offtake can be simplified as follows:

$$\frac{\partial(UA)}{\partial X} = 0 \qquad \text{.......................} \quad (1)$$

$$\frac{\partial(U^2A)}{\partial X} + gA\frac{\partial h}{\partial X} = gA(S_x - S_f) \qquad \text{.......................} \quad (2)$$



CHANNEL CROSS SECTION

Cross sectional area $A = h(b + \beta h)$

Wetted perimeter $P = (b + 2\sqrt{1 + \beta^2}h)$

Hydraulic radius $R = \frac{A}{P} = \frac{h(b + \beta h)}{(b + 2\sqrt{1 + \beta^2}h)}$

$$(1) \quad U\frac{\partial A}{\partial X} + A\frac{\partial U}{\partial X} = 0 \qquad \text{.......................} \quad (3)$$

Multiplying (3) by

$$U^2\frac{\partial A}{\partial X} + UA\frac{\partial U}{\partial X} = 0 \qquad \text{.......................} \quad (4)$$

$$(2), \quad U\frac{\partial(UA)}{\partial X} + UA\frac{\partial U}{\partial X} + gA\frac{\partial h}{\partial X} = gA(S_x - S_f) \qquad \text{..........} \quad (5)$$

Substitute (1) and (4) in (5) gives

$$-U^2\frac{\partial A}{\partial X} + gA\frac{\partial h}{\partial X} = gA(S_x - S_f)$$

Substitute value of A

$$-U^2\frac{\partial}{\partial X}[h(b + \beta h)] + gh(b + \beta h)\frac{\partial h}{\partial X} = gh(b + \beta h)(S_x - S_f)$$

$$-U^2[b\frac{\partial h}{\partial X} + 2\beta h\frac{\partial h}{\partial X}] + gh(b + \beta h)\frac{\partial h}{\partial X} = gh(b + \beta h)(S_x - S_f)$$

Simplifying

$$\frac{\partial h}{\partial X} = \frac{gh(b + \beta h)(S_x - S_f)}{gh(b + \beta h) - U^2(b + 2\beta h)}$$

$$\frac{\partial h}{\partial X} = \frac{(S_x - S_f)}{1 - \frac{U^2(b + 2\beta h)}{gh(b + \beta h)}}$$

Using Mannings Equation:

$$U = \frac{1}{n}R^{2/3}S_f^{1/2}$$

Substituting value of R and U in above equation:

$$S_f = \frac{Q^2 n^2 (b + 2\sqrt{1 + \beta^2}h)^{4/3}}{h^{10/3}(b + \beta h)^{10/3}}$$

Therefore expression for $\frac{\partial h}{\partial x}$ could be written as follows

$$\frac{\partial h}{\partial x} = \frac{S_x - \frac{Q^2 n^2 (b + 2\sqrt{1 + \beta^2}h)^{4/3}}{h^{10/3}(b + \beta h)^{10/3}}}{1 - \frac{Q^2 (b + 2\beta h)}{gh^3 (b + \beta h)^3}} \qquad \text{.......................} \quad (6)$$

The change in $\frac{\partial h}{\partial x}$ is mostly governed by h3 and integration is carried out by keeping other exprcssions constant

$$\text{Let } \mu = \frac{Q^2\, n^2\, (b + 2\sqrt{1+\beta^2}h)^{4/3}}{(b + \beta h)^{10/3}\, h^{1/3}}$$

$$\Omega = \frac{Q^2\, (b + 2\beta h)}{g\, (b + \beta h)^3}$$

Hence:

$$\frac{\partial h}{\partial x} = \frac{S_x - \frac{\mu}{h^3}}{1 - \frac{\Omega}{h^3}}$$

$$\int_{h_c}^{h} \frac{(h^3 - \Omega)}{(S_x h^3 - \mu)}\, dh = \int_{L}^{X} dx$$

Expressing dh as dh^3:

$$\int_{h_c}^{h} \frac{(h^3 - \Omega)}{(S_x h^3 - \mu)}\, dh^3 = \int_{L}^{X} 3h^2 dx$$

$$\int_{h_c}^{h} [-1 + \frac{\frac{\mu}{S_y} - \Omega}{\frac{\mu}{S_x} - h^3}]\, dh^3 = -\int_{L}^{X} 3S_x h^2\, dx$$

$$\{-h^3 + h_C^3 - (\frac{\mu}{S_x} - \Omega) \ln \frac{(\frac{\mu}{S_x} - h^3)}{(\frac{\mu}{S_x} - h_C^3)}\} = 3S_x h^2 (L - X)$$

Finally rcarranging the terms X could be express as function of h as follows :

$$X = L + \frac{1}{3S_x h^2}(h^3 - h_C^3) + \frac{1}{3S_x h^2}(\frac{\mu}{S_x} - \frac{Q^2(b + 2\beta h)}{(b + \beta h)^3}) \ln \frac{(\frac{\mu}{S_x} - h^3)}{(\frac{\mu}{S_x} - h_C^3)} \quad \ldots (7)$$

$$\text{where } \mu = \frac{Q^2\, n^2\, (b + 2\sqrt{1+\beta^2}h)^{4/3}}{(b + \beta h)^{10/3}\, h^{1/3}}$$

The Numerical solution based on fourth order Runge-Kutta method is compared with the above analytical solution for the following case.

Length of canal (L) = 1000 m
Channel Discharge (Q) = 0.70 m^3/sec
Channel bed slope (S_x) = 0.00035
Channel side slopes (β) = 1.5

Downstream boundary condition : Critical flow

The water surface profile obtained from analytical and numerical method is shown in Figure 1.

3.0 Analysis 2: Analytical solution for water surface profile with offtake outflow

Under steady state the continuity and momentum equations in open channel with lateral outflow of q1 per unit length from an offtake could be written as follows :

$$\frac{\partial(UA)}{\partial X} = -q_l \quad \ldots\ldots (8)$$

$$\frac{\partial(U^2A)}{\partial X} + gA\frac{\partial h}{\partial X} = gA(S_x - S_f) \quad (9)$$

Downstream component of lateral outflow (since in most of the cases the direction of outflow is perpendicular to the channel flow) is neglected in the momentum equation.

$$(8)\quad U\frac{\partial A}{\partial X} + A\frac{\partial U}{\partial X} = -q_l \quad \ldots\ldots (10)$$

Multiflying (10) by U,

$$U^2\frac{\partial A}{\partial X} + UA\frac{\partial U}{\partial X} = -Uq_l \quad \ldots\ldots (11)$$

$$(9).\quad U\frac{\partial(UA)}{\partial X} + UA\frac{\partial U}{\partial X} + gA\frac{\partial h}{\partial X} = gA(S_x - S_f) \quad \ldots\ldots (12)$$

Substitute (8) and (11) in (12) gives

$$-U^2\frac{\partial A}{\partial X} + gA\frac{\partial h}{\partial X} = gA(S_x - S_f) + 2Uq_l$$

Substitute value of A

$$-U^2\frac{\partial}{\partial X}[h(b+\beta h)] + gh(b+\beta h)\frac{\partial h}{\partial X} = gh(b+\beta h)(S_x - S_f) + 2Uq_l$$

$$-U^2[b\frac{\partial h}{\partial X} + 2\beta h\frac{\partial h}{\partial X}] + gh(b+\beta h)\frac{\partial h}{\partial X} = gh(b+\beta h)(S_x - S_f) + 2Uq_l$$

Simplifying

$$\frac{\partial h}{\partial X} = \frac{gh(b+\beta h)(S_x - S_f) + 2Uq_l}{gh(b+\beta h) - U^2(b+2\beta h)}$$

$$\frac{\partial h}{\partial X} = \frac{(S_x - S_f) + \frac{2Uq_l}{gh(b+\beta h)}}{1 - \frac{U^2(b+2\beta h)}{gh(b+\beta h)}}$$

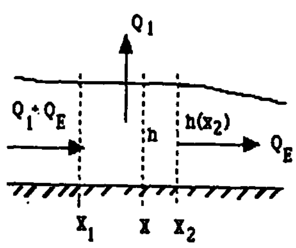
Using Mannings Equation:

$$U = \frac{1}{n}R^{2/3}S_f^{1/2}$$

Substituting value of R in above equation;

$$S_f = \frac{U^2 n^2 (b + 2\sqrt{1+\beta^2}h)^{4/3}}{h^{4/3}(b+\beta h)^{4/3}}$$

$$\frac{\partial h}{\partial x} = \frac{S_x - \frac{U^2 n^2 (b + 2\sqrt{1+\beta^2}h)^{4/3}}{h^{4/3}(b+\beta h)^{4/3}} + \frac{2Uq}{gh(b+\beta h)}}{1 - \frac{U^2(b+2\beta h)}{gh(b+\beta h)}} \quad \text{.............(13)}$$



Q_E = Channel discharge after the offtake.
Q_1 = Offtake discharge
X_1 = Starting location of the offtake
X_2 = Ending location of the offtake
$X_2 - X_1$ = Width of the offtake

Mean velocity any location X and lateral outflow per unit width q_l are given by

$$U = \frac{[Q_E + \frac{Q_1(X_2 - X)}{(X_2 - X_1)}]}{h(b+\beta h)} \quad \text{and} \quad q_l = \frac{Q_1}{(X_2 - X_1)}$$

or $U = B + CX$

$$\text{where} \quad B = \frac{[Q_E + \frac{Q_1 X_2}{(X_2 - X_1)}]}{h(b+\beta h)}$$

$$C = \frac{-Q_1}{(X_2 - X_1)h(b+\beta h)}$$

$$\frac{\partial h}{\partial X} = \frac{(S_x - S_f)}{1 - \frac{U^2(b+2\beta h)}{gh(b+\beta h)}} + \frac{\frac{2Uq_l}{gh(b+\beta h)}}{1 - \frac{U^2(b+2\beta h)}{gh(b+\beta h)}}$$

$$h = h(X_2) + \int_{X_2}^{X} \frac{(S_x - S_f)}{1 - \frac{U^2(b+2\beta h)}{gh(b+\beta h)}}dX + \int_{X_2}^{X} \frac{\frac{2Uq_l}{gh(b+\beta h)}}{1 - \frac{U^2(b+2\beta h)}{gh(b+\beta h)}}dX \quad \text{.............(14)}$$

Since $(S_x - S_f)$ is so small, the change in h is mostly governed by the lateral outflow component (second integral). Hence first integral is neglected in comparison to second integral.

$$h = h(X_2) + \int_{X_2}^{X} \frac{\frac{2Uq_l}{gh(b+\beta h)}}{1 - \frac{U^2(b+2\beta h)}{gh(b+\beta h)}}dX \quad \text{.............(15)}$$

$$h = h(X_2) + \int_{X_2}^{X} \frac{\frac{2(B+CX)\frac{Q_1}{(X_2 - X_1)}}{gh(b+\beta h)}}{1 - \frac{(B+CX)^2(b+2\beta h)}{gh(b+\beta h)}}dX \quad \text{.............(16)}$$

$$A = \frac{2Q_1}{(X_2 - X_1)gh(b+\beta h)} \; ; \quad D = \frac{(b+2\beta h)}{gh(b+\beta h)}$$

$$h = h(X_2) + \int_{X_2}^{X} \frac{A(B+CX)}{1 - D(B+CX)^2}dX$$

$$h = h(X_2) - \frac{A}{2DC}\ln\left[\frac{1 - D(B+CX)^2}{1 - D(B+CX_2)^2}\right]$$

Then X as function of h is given by following formula.

$$X = -\frac{B}{C} + \frac{1}{C\sqrt{D}}[1 - [(1 - D(B+CX_2)^2 \exp\frac{2DC}{A}(h(X_2) - h))]^{0.5}$$

The Numerical solutions based on fourth order Runge-Kutta method is compared with the above Analytical solution for the following cases.

Length of Offtake $(X_2-X_1) = 2m; X_2=2m; X_1=0m$
Offtake discharge $(Q_1) = 0.06, 0.10, 0.20$ m3/sec
Channel discharge below offtake $(Q_E) = 0.70\ m^3/sec$
Channel bed slope $(S_x) = 0.00035$
Downstream boundary condition is known
i.e. $h(X_2) = 0.6003$ m (from previous analysis)

4.0 Discussion

The water surface profile near the offtake for all three cases are shown in Figures 2, 3 and 4 respectively.

Offtake discharge (Q1) (m^3/sec)	Down stream depth $h(X_2)$ (m)	Up stream depth $h(X_1)$ (m)
0.06	0.6003	0.5792
0.10	0.6003	0.5949
0.20	0.6003	0.5884

As per analysis 1.0 the water surface profile gradually rises from down stream to up stream of the canal (Figure 1). However, as per analysis 2.0, the water surface profile near the offtake (Figures 2, 3 & 4) drops suddenly along the width of the offtake proceeding from down stream towards up stream. The down stream boundary condition for analysis 1.0 is $h=h_c$, i.e. critical flow condition and for the analysis 2.0, $h=h(X_2)$, i.e. is the depth of water at the down stream of the offtake. The water surface profile, up stream of the offtake could be obtained in a similar manner by repeating the same procedure in analysis 1.0 with the down stream boundary condition of $h=h(X_1)$ and it should have the same pattern of rising towards up stream to down stream. The interesting feature observed in analysis is the abrupt change in water depth near the offtake, i.e. slight increase in water depth in the direction of flow along the width of offtake due to momentum change.

5.0 Conclusions

The analytical solutions obtained for water surface profiles give more accurate results for both conditions of with and without offtake outflow as compared to numerical results. As such analytical equations can be used in a main system hydraulic model with a reasonable accuracy. Another advantage of analytical solutions is the removal of the singularity that exist in water surface gradients under critical flow condition where the solution becomes difficult as the gradient becomes theoreticaly infinite. Further, for solutions by numerical methods a more accurate boundary condition can be obtained by analytical equations under critical flow condition. One of the interesting phenomena observed by the analysis is that down stream water depth is slightly higher than up stream water depth near the offtake and there is a gradual rise in water surface due to momentum change in removal of water.

References :

Godaliyadda, G.G.A. (1990) "Improvement on irrigation main system management in Sri Lanka with emphasis on water conveyance and distribution", Ph.D. research proposal submitted to Cornell University, Ithaca, New York, U.S.A.

Liggett, J.A. (1975) "Basic equations of unsteady flow", Chapter 2, Unsteady flow in open channels, Volume 1, Edited by K. Mahamood and V. Yevjevich.

Liggett, J.A. and Cunge, J.A. (1975) "Numerical methods of solution of the unsteady flow equations", Chapter 4, Unsteady flow in open channels, Volume 1, Edited by K. Mahamood and V. Yevjevich.

Figure 1: Water surface profile

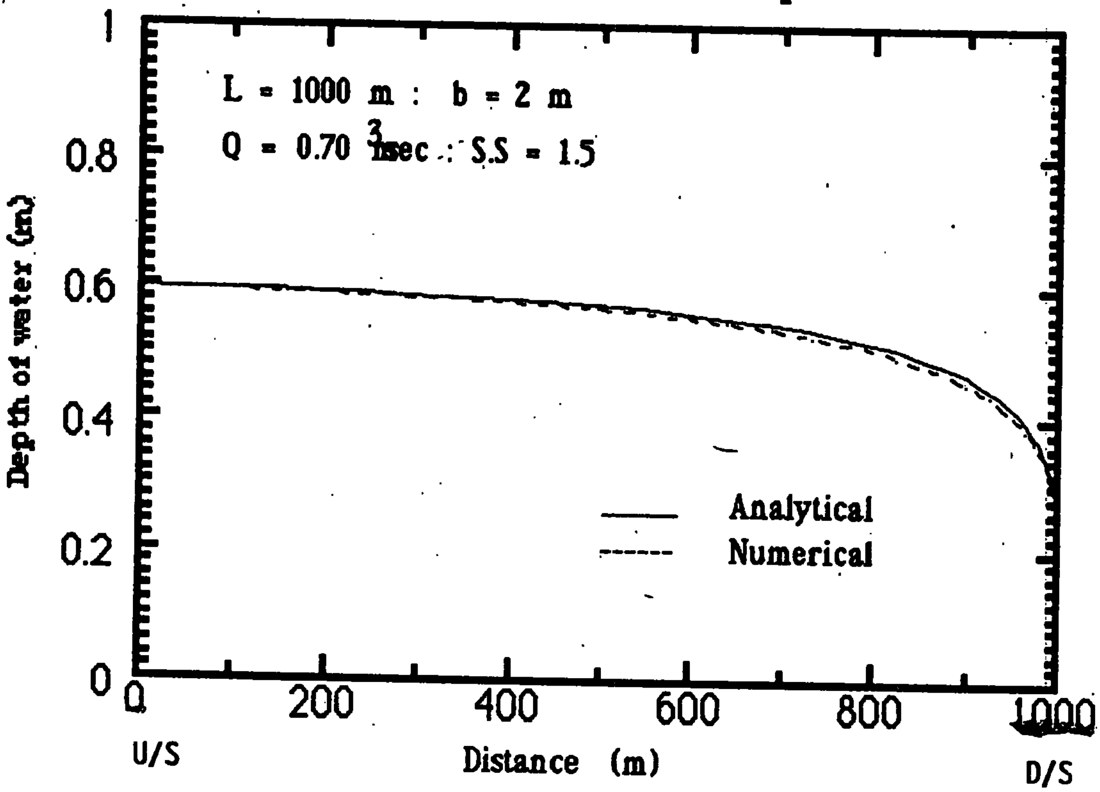


Figure 2: Water profile near offtake

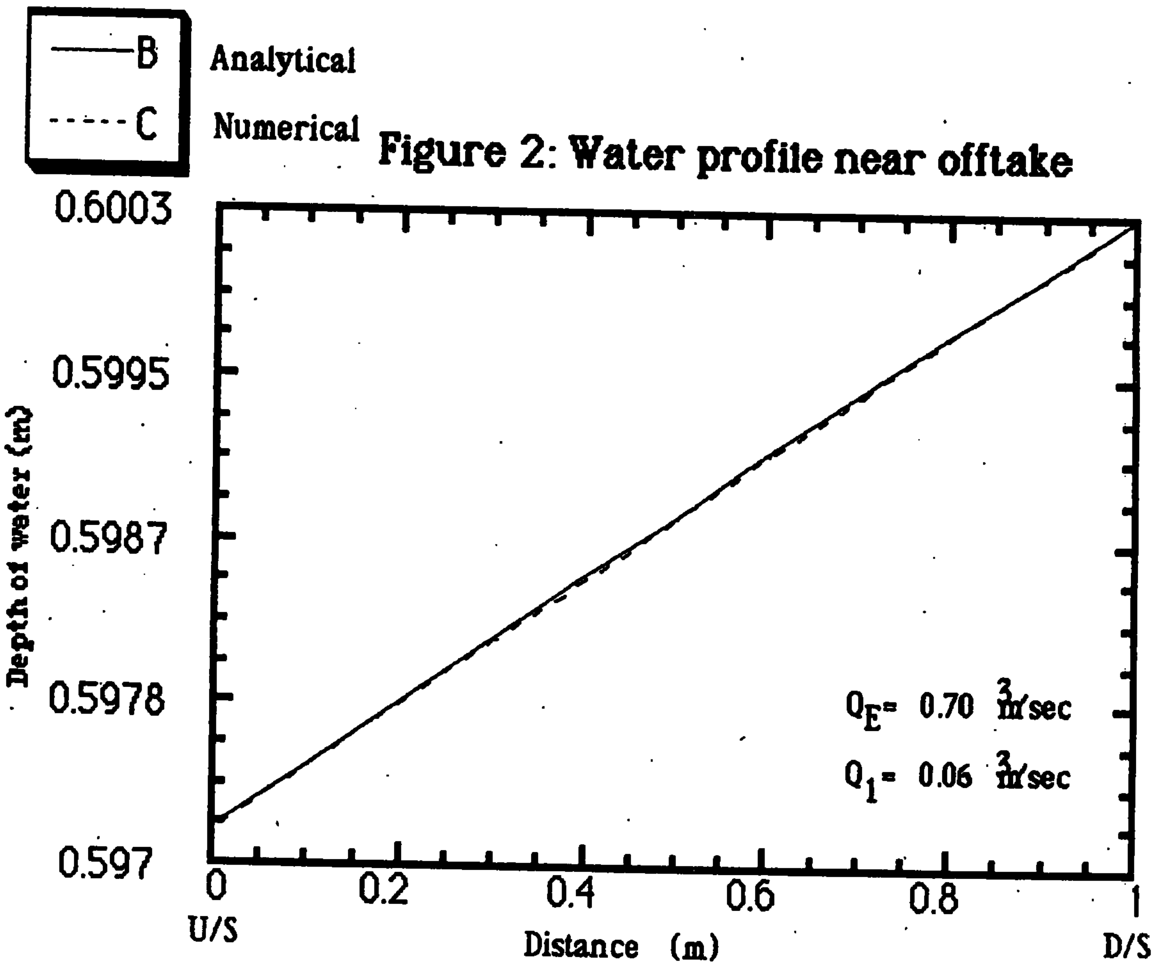


Figure 3: Water profile near offtake

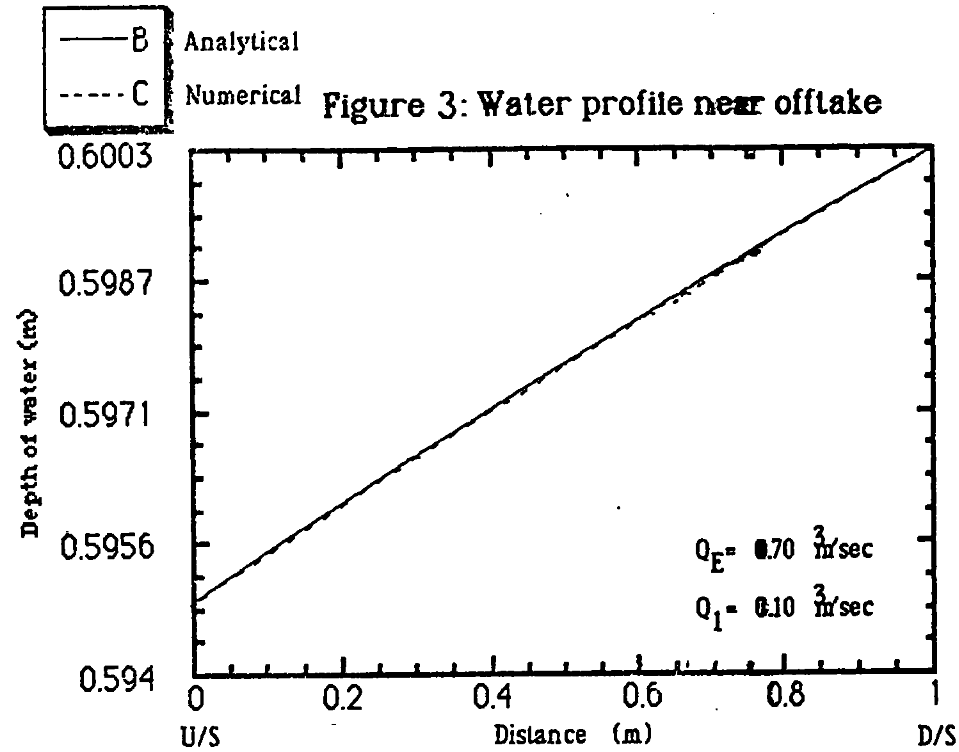
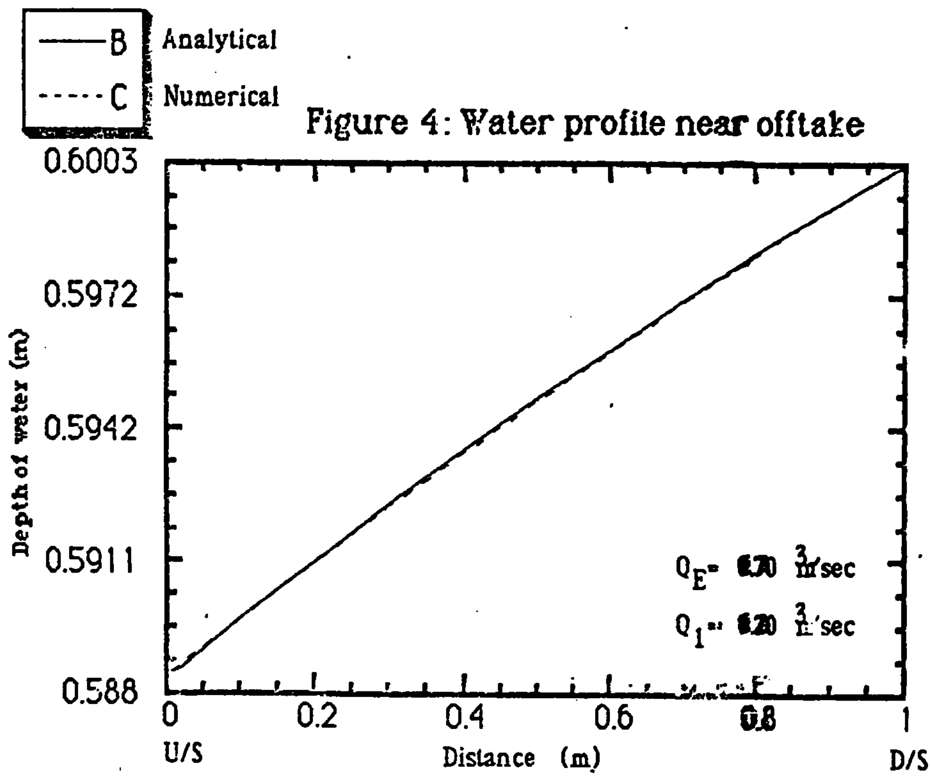


Figure 4: Water profile near offtake



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