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A MATHEMATICAL MODEL FOR THE STUDY OF MASS OSCILLATIONS IN SURGE TANKS

by

A.K.D.N. Atukorala

1.0 Introduction

Surge tanks are employed in Hydro-Electric or large pipeline systems to prevent both objectionably high and low pressure surges generated by transients in the system. Graphical methods of waterhammer analysis have been used in the past in designing of surge tanks with generally good results, but using digital computers the accuracy of results and the speed of analysis can be improved. The elastic wave equation can be solved by using the Method of Characteristics on a digital computer. The object of this program is to use the Four Point Method of Characteristics to analyse the elastic wave equation. It is assumed that the surge tank acts as a temporary storage for excess water which has been diverted by the main flow due to high pressures; or acts as a source of supply water when pressure drops. This means that the dynamic effect of the water column in the surge tank is neglected. The surge tank acts as an internal boundary to the system and at this location the Three Point Method of Characteristics is used. This program is written for the restricted orifice type surge tank but can be easily extended to the simple and the differential types of surge tanks. The particular advantages of this method are that friction and the boundary conditions can conveniently be included.

2.0 Theoretical Considerations

Consider the pipe line shown in Fig. 1 which is self explanatory. The condition of dynamic equilibrium can be set up for an element dx along the pipe line as shown.

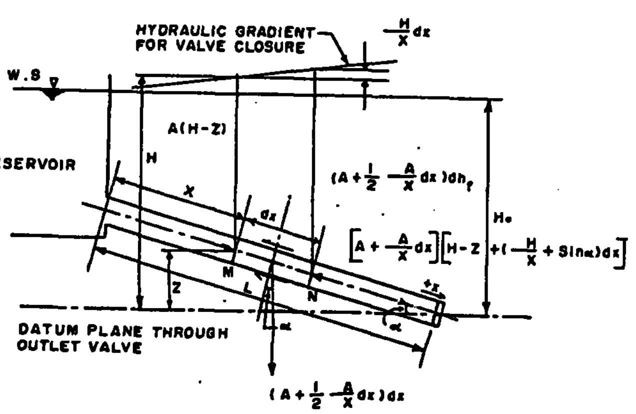

Fig. 1

The governing equations which provide the basis for the analysis of Waterhammer phenomena are the momentum equation,

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial H}{\partial x} + f.V.|V|/2.D = 0 \qquad (1)$$

and the continuity equation

$$\frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x} + \frac{a2}{g}\frac{\partial V}{\partial x} = 0 \qquad (2)$$

In the equations 1 and 2, H denotes the piezometric head at location 'x' and time 't', V is the average velocity of flow, D is the diameter of pipe, f is the friction factor in Darcy-Weisbach formula, x is the distance along the center-line of the pipe, g is the gravitational constant and 'a' is the celerity of the pressure surge.

2.1 Solution of Governing Equations

To obtain the pressure head and velocity changes with time 't' along the length 'x' the above two equations are solved numerically with the total derivatives for pressure head and velocity. The finite difference approximation is used as the principle for the numerical solution of the above equations.

The equations 1 and 2 constitute a Hyperbolic system and the characteristic relations can be obtained from the following set of equations.

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$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial H}{\partial x} = \frac{f.V.}{2.D}|V|$$

$$\frac{a^2}{g}\frac{\partial V}{\partial x} + \frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x} = 0$$

$$dt\frac{\partial V}{\partial t} + dx\frac{\partial V}{\partial x} = dV$$

$$dt\frac{\partial V}{\partial t} + dx\frac{\partial H}{\partial x} = dH$$

This set of equations can be rearranged in the following matrix form

$$\begin{vmatrix} 1 & V & 0 & g \\ 0 & a^2/g & 1 & V \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{vmatrix} \begin{vmatrix} \partial V/\partial t \\ \partial V/\partial x \\ \partial H/\partial t \\ \partial H/\partial x \end{vmatrix} = \begin{vmatrix} -f.V.|V|/(2.D) \\ 0 \\ dV \\ dH \end{vmatrix} \qquad (3)$$

The directions of the two families of characteristic curves of C+ and C- are given by the condition that the determinant

$$\begin{vmatrix} 1 & V & 0 & g \\ 0 & a^2/g & 1 & V \\ dt & dx & 0 & 0 \\ 0 & 0 & dt & dx \end{vmatrix} = 0$$

Which gives $dx/dt = V \pm a$

The ordinary differential equations along the characteristics can be obtained by replacing any one column of the above-determinant by the R.H.S. of equation (3).

Hence the relations along the characteristic directions are given by

$$\begin{vmatrix} 1 & V & 0 & -f.V.|V|/(2.D) \\ 0 & a^2/g & 1 & 0 \\ dt & dx & 0 & dV \\ 0 & 0 & dt & dX \end{vmatrix} = 0$$

which gives the following solution

$$\frac{dV}{dt} + g\left(\frac{dx}{dt} - V\right).\frac{dH}{dt} + \frac{f.V.|V|}{2.D} = 0$$

Along the C+ characteristic, $dx/dt = V + a$

and $\frac{dV}{dt} + \frac{g}{a}.\frac{dH}{dt} + \frac{f.V}{2.D}|V| = 0$

Along the C- characteristic, $dx/dt = V - a$

and $\frac{dV}{dt} - \frac{g}{a}\frac{dH}{dt} + \frac{f.V.}{2.D}|V| = 0$

2.2 Assumptions

It is assumed that the velocity and pressure head are uniform over any cross section along the penstock and tunnel and both remain full of water with sufficient minimum pressure to exceed the vapour pressure. The water level at the reservoir is assumed to be constant during the transient period. It is also assumed that the surface area of the reservoir is so large that the inflow into the reservoir or the flow due to the transient does not change the reservoir level during the transient period.

The characteristic directions of the governing equations are given by

$dx/dt = V \pm a$

The celerity of the pressure wave 'a' is approximately equal to the velocity of sound in water and the average flow velocity 'V' is very much smaller when compared with 'a'. Therefore we can assume that the characteristic directions are approximately given by

$dx/dt = \pm a$

which are straight lines provided the value of 'a' is a constant. It is also assumed that the surge tank acts as temporary storage for excess water which has been diverted from the mainflow due to the high pressures, and it acts as a source of supply of water when the pressure drops. This means that the dynamic effect of the water column in the surge tank is neglected.

The average flow velocity and the pressure head in the pipe varies with time. The Darcy's friction factor and the discharge coefficient for the orifice depend on the Reynold's Number characterising the instantaneous flow. But for simplicity it is assumed that these coefficients are constants throughout the calculations.

2.3 Four Point Method of Characteristics

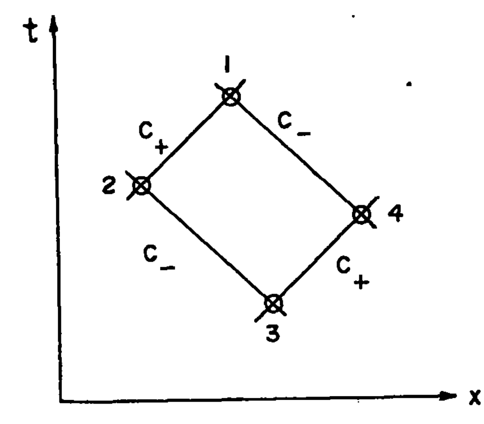

Fig. 2

Consider an element bounded by two C+ and C- characteristics as shown in Fig. 2. The points of intersection of these characteristics are denoted by the integers 1,2,3 and 4 reckoned anticlockwise as shown.

The direction of characteristics are such that

for C+ characteristics $dx/dt = a$

and for C- characteristics $dx/dt = -a$

Since the characteristics are straight lines the following equations can be written.

$$
\begin{aligned}
x_1 - x_2 &= a(t_1 - t_2) \\
x_4 - x_3 &= a(t_4 - t_3) \\
x_2 - x_3 &= a(t_2 - t_3) \\
x_1 - x_4 &= a(t_1 - t_4)
\end{aligned}
$$

By solving these equations for x and t, we get

$$x_1 = x_2 + x_4 - x_3$$

and $$t_1 = t_2 + t_4 - t_3$$

The ordinary differential equations along the characteristics are

$$C+ \quad \frac{dV}{dt} + \frac{g}{a}\frac{dH}{dt} + \frac{f.V.|V|}{2.D} = 0$$

$$C- \quad \frac{dV}{dt} - \frac{g}{a}\frac{dH}{dt} + \frac{f.V.|V|}{2.D}$$

Using finite differences for the derivatives, the following equations can be written for the above element bounded by characteristics shown in Fig. 2.

$$V_1 - V_2 + g(H_1 - H_2)/a + (t_1 - t_2).f.V_1.\ |V_2|\ /2.D = 0$$

$$V_2 - V_3 + g(H_2 - H_3)/a + (t_2 - t_3).f.V_2.\ |V_3|\ /2.D = 0$$

$$V_4 - V_3 + g(H_4 - H_3)/a + (t_4 - t_3).f.V_4.\ |V_3|\ /2.D = 0$$

$$V_1 - V_4 + g(H_1 - H_4)/a + (t_1 - t_4).f.V_1.\ |V_2|\ /2.D = 0$$

These four equations can be solved to obtain the following values for V_1 and H_1.

$$V_1 = [V_2 + V_4 - V_3 + (f/4D).|V_3|[V_2(t_2 - t_3) + V_4(t_4 - t_3)]]/ [1 + (f/4D).[(t_1 - t_2).|V_2| + (t_1 - t_4).|V_4|]]$$

and

$$H_1 = H_2 + H_4 - H_3 + (f/4D).[(t_1 - t_4).V_1.|V_4| + (t_4 - t_3).V_4.|V_3| - (t_2 - t_3).V_2.|V_3| - (t_1 - t_2).V_1|V_2|]$$

2.4 Initial and Boundary Conditions

The initial conditions are of two point characteristic nature corresponding to conditions that are initial in the sense of time.

The application of the four point method of characteristics necessitates the construction of the first time step from the initial conditions.

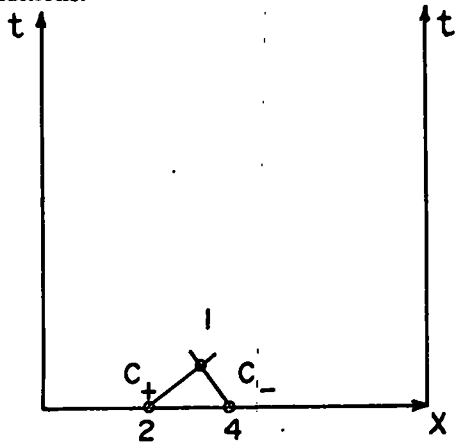

Fig. 3

For the data along the line $t = O$ we can write

$$x_1 - x_2 = a(t_1 - t_2)$$

and

$$x_1 - x_4 = -a(t_1 - t_4)$$

which gives

$$t_1 = 0.5(x_4 - x_2 + t_2 + t_4)$$

and

$$x_1 = 0.5\ [x_2 + x_4 + a(t_4 - t_2)]$$

Since the initial condition $t_2 = t_4 = O$

Then

$$t_1 = (x_4 - x_2)/2a$$

and

$$x_1 = (x_2 + x_4)/2$$

Along characteristic lines we can write

$$V_1 - V_2 + g(H_1 - H_2)/a + t_1.f.V_1.\ |V_2|\ /2.D = 0$$

and

$$V_1 - V_4 + g(H_1 - H_4)/a + t_1.f.V_1.\ |V_4|\ /2.D = 0$$

Solving above two equations for V_1 and H_1, we get

$$V_1 = [V_2 + V_4 + (g/a).\ (H_2 - H_4)/[2 + (f.t_1/2D)(\ |V_2| + |V_4|\ )]$$

and

$$H_1 = 0.5\{H_2 + H_4 + a[V_2 - V_4 - f.t_1.V_1(\ |V_2|\ .\ |V_4|\ )/2.D]/g\}$$

The one point boundary conditions along a line x = constant is given at the L.H.S. and R.H.S. boundaries.

2.4.1 L.H.S. Boundary (at the valve)

If the velocity is known to vary in some prescribed manner at the valve then an expression for the head can be found.

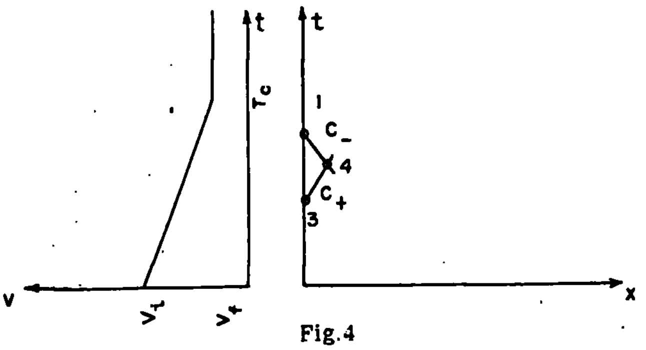

Fig.4

Suppose that the valve is closed or opened in a manner which causes the velocity to decrease or increase linearly from V_1 to V_f in time T_c sec.

Then $V_1 = V_f + (V_1 - V_f).(T_c - t1)/Tc$ $\quad (0 \geq t_c < T_c)$

and $V_1 = V_f$ $\quad (t_1 \geq T_c)$

Therefore along the C- characteristics

$$V_1 - V_4 - g(H_1 - H_4)/a + (t_1 - t_4).f.V_1.\ |V_4|\ /2.D = 0$$

This follows that

$$H_1 = H_4 + a[V_1 - V_4 + (t_1 - t_4).f.V_1.\ |V_4|\ /2.D]/g$$

2.4.2 R.H.S. Boundary (At the reservoir)

According to the assumption that the water level in the reservoir does not change during the period of simulation, it can be considered that the water level at the reservoir is constant. Therefore the value of H remains constant for all time. (neglecting the velocity head).

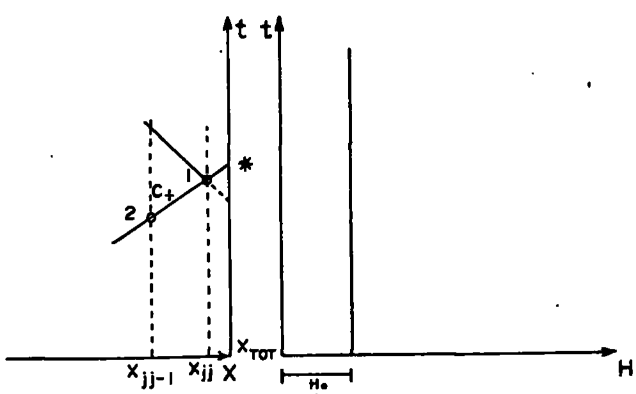

Fig. 5

Let X_{TOT} be the x-coordinate at the reservoir and x_{jj} be the last grid point on x-axis. Let these two points not coincide.

For the two points (2) and (*) along the C+ characteristic

$$V_* - V_2 + g(H_* - H_2)/a + (t_* - t_2).f.V_*.\ |V_2|\ /2D = 0$$

$$X_{TOT} - X_2 = a(t_* - t_2)$$

At the boundary $H_* = H_0$ = Head of reservoir level

Hence the value of V* can be given as follows.

$$V_* = [V_2 - g(H_0 - H_2)/a]/[1+(t_* - t_2).f.\ |V_2|\ /2.D]$$

Assuming that the velocity and pressure head varies linearly between points (2) and (*) the velocity and pressure head at the grid point (l) can be written as follows.

$$V_1 = V_2 + (V_* - V_2).(X_1 - X_2)/(X_{TOT} - X_2)$$

and

$$H_1 = H_2 + (H_* - H_2).(X_1 - X_2)/(X_{TOT} - X_2)$$

2.4.3 Internal Boundary Conditions (At surge tank)

To incorporate the surge tank into the water hammer analysis it is necessary to treat it as a special internal boundary condition. Due to a sudden closure or opening of the downstream valve pressure waves are propagated and travel along the penstock and the tunnel. This causes a rapid change of pressure and velocity with time within the system. Initially the water level at the surge tank is at dynamic or static equilibrium depending on the flow condition in the system.

As the pressure head in the tunnel at the surge tank junction varies with time, a pressure difference develops between the tunnel and surge tank. On account of this pressure difference water tends to flow in to or out of the surge tank. At the junction of the surge tank the continuity and the orifice equations are incorporated into the waterhammer boundary conditions represented by equations (1) and (2).

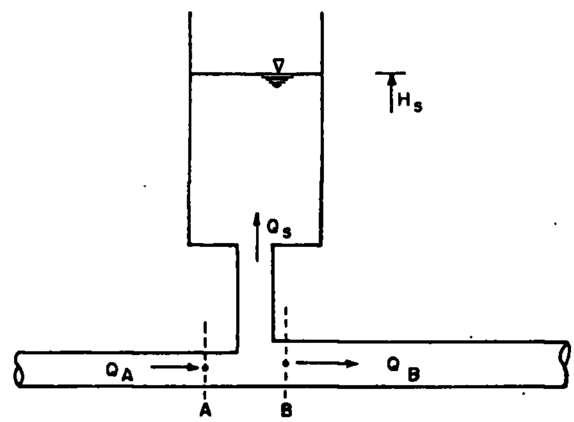

Fig. 6

The appropriate characteristic equations for points A and B are

$$V_A - V_2 + g(H_A - H_2)/a_A + (t_A - t_2).f_A.V_A.\ |V_2|\ /2.D_A = 0$$

$$V_B - V_4 - g(H_B - H_4)/a_B + (t_B - t_2).f_B.V_B.\ |V_4|\ /2.D_B = 0$$

Since the values of V2, V4, H2 and H4 are known the above equations can be rearranged in the following form;

$$V_A = C_{1A}H_A + C_{2A} \qquad (5)$$

$$V_B = C_{1B}H_B + C_{2B} \qquad (6)$$

where

$$C_{1A} = g/a_A[1 + f_A(t_A - t_2).\ |V_2|\ /2.DA] \quad \text{and}$$

$$C_{2A} = (V_2 + gH_2/a_A)/[1 + f_A(t_A - t_2).\ |V_2|\ /2.DA]$$

$$C_{1B} = g/a_B[1 + f_B(t_B - t_4).\ |V_4|\ /2.D_B] \quad \text{and}$$

$$C_{2B} = (V_4 + gH_4/a_B)/[1 + (f_B/2D)(t_B - t_4).\ |V_4|\ /2.D_B]$$

Neglecting the dynamic effect within the control volume between sections (A) and (B) including the surge tank free surface,

the pressures at A and B can be assumed as equal.

i.e. $H_A = H_B = H_P \qquad (7)$

The discharge into the surge tank Qs is given by the continuity equation;

$$Qs = Q_A - Q_B \qquad (8)$$

$= A_A.V_A - A_B.V_B$ where and represent respective cross sectional areas.

If the static head of the surge tank is H_s^n (after 'n' time steps) then the flow through the orifice can be written in the following form;

$$Q_s^n = C_d A_T/2g(H_p^n - H_s^n) \qquad (9)$$

where C_d and A_T are the discharge coefficient and the cross sectional area of the orifice respectively.

The new height of the water level in the surge tank at the end of the time step 'n' is given by

$$H_s^{n+1} = H_s^n + Q_s^n\ t/As \qquad (10)$$

where $_t$ and A_s are the time lag between time step 'n' and 'n+1', and cross sectional area of the surge tank, respectively.

The above equations (5) to (10) contain six unknowns namely V_A, V_B, H_A, H_B, Q_S and H_S can be solved.

The computer program is based on the equations derived under the numerical solution to the partial differential equations for the momentum and continuity and the boundary conditions mentioned above. It is important that the time increments in both penstock and tunnel to be equal the x increments could not be equal. Hence the number of grid points on penstock is given as input data and the number of grid points on tunnel is calculated accordingly.

2.4.4 Computer Program

This program is written according to the theory and initial conditions discussed above. Even though this program is written specially for a surge tank it can be extended to calculate waterhammer pressures in pipelines. As part of the output of this program the pressures and velocities at each grid point can be obtained for each time step but this is omitted since it needs more computer time and space. Therefore the output is limited to the water level variations and flow velocities at upstream and downstream of the surge tank at every 10th time step.

Main Program

The main program reads the input data from an input file through the Subroutine READER and makes basic calculations within the main program. Then it calls for different sub programs to calculate the different steps in calculations. Finally the results are printed into an outfile named "RESULTS".

Since the time steps are very small (generally) in the range of 0.15 to 0.30 sec.) it is necessary to run the program for a large number of time steps required for a simulation period of 1800 sec. will be approximately 10,000. As the computer capacity is limited it is programmed such that it initially runs for a smaller time period and the output of the last time step is fed back to the program as input to continue for the next time period. This procedure is repeated until the required simulation time is covered.

Sub Programs

1. Prelim

This sub program calculates the preliminary stage of the first time step from the initial pressures and velocities by using the three point method.

2. Inter

This subprogram calculates the values of pressure and velocity at each grid point starting from time step two by using the four point method of characteristics.

3. Great

This subprogram represents the calculations of the surge tank water levels using the six equations derived under the internal boundary conditions.

4. Boundl

This subprogram treats the boundary conditions at the valve. Since the boundary condition at the reservoir is so simple it is incorporated in the main program.

5. Chrac

This subprogram selects the appropriate subprogram to be used for the calculations at the new point depending on the position of the grid point.

2.4.5 Input Data

The data representing the geometrical characteristics of the penstock and the tunnel are read into the program. The physical properties such as elastic modulus, friction factors, discharge coefficients are also given as input data. The initial and boundary conditions, operation time of valve and period of simulation is also included in the input data. In addition to above data the number of grid points selected along the penstock is also required to proceed with the calculations.

The program was run for the following input data which represent the approximate values of a Hydro-Power scheme in Sri Lanka in construction of which the writer was involved, and the field measurements of water level oscillations during the load rejection tests of the power plant.

Length of headrace tunnel	5600.0	M
Length of penstock	800.0	M
Diameter of headrace tunnel	6.1	M
Diameter of penstock (equivalent)	5.37	M
Diameter of surge tank	20.0	M
Diameter of surge tank orifice	2.0	M
Elevation of reservoir water level	438.0	M
Elevation of tunnel intake	355.0	M
Elevation of surge tank position	299.0	M
Elevation of valve	232.0	M

Following physical properties were obtained from the references mentioned at the end of the text.

Specific weight of water	1000.0	Kg/Cu.M
Bulk modulus of Elasticity of water	20.7×10^{8}	N/Sq.M
Young's modulus of steel	20.7×10^{10}	N/Sq.M
Modulus of rigidity of rock	9.58×10^{9}	N/Sq.M
Darcy's friction factor of steel	0.0143	
Darcy's friction factor of concrete	0.0183	
Coefficient of discharge (Inflow)	0.74	
Coefficient of discharge (Outflow)	0.91	

Number of grid points on penstock	2
Operation time of valve	varies
Simulation time	varies

In the data the elevations are given with respect to the Mean Sea Level. The valve operation time and the simulation time are different for each test.

2.4.6 Results

The following test runs were made and the graphical computer outputs are attached (Figs 6a-6f)

(i) Static test
(ii) Steady state test
(iii) Sudden closure of valve (Closing time 5 sec.)
(iv) Partial closure of valve (Closing time 5 sec.)
(v) Sudden opening of valve (Opening time 10 sec.)
(vi) Sudden closure of valve without an orifice (Closing time 5 sec.)

3.0 Discussion

The static and the steady state tests were simulated for a period of 1200 sec. The static test gives constant water level throughout the simulation time which equals the reservoir level. In the steady state test 1.0mm difference in water level is observed after 245 sec. (866 time steps) and 0.174m. difference in water level is observed after 1200 sec. (4237 time steps). This deviation from the exact solution can be due to the following two reasons.

(i) The truncation errors due to the assumption of $dx/dt = a$ and the errors in the finite difference approximations of the derivatives.

(ii) The rounding off of errors in the numerical solution influenced by the limited numerical accuracy of the computer.

Both these type of errors occur at each time step of calculations and their combined effect is propagated through the solution to give a cumulative error.

Generlly the maximum surge occurs in between 150 to 400 sec. and the numerical errors in this time period are very small. Therefore the results obtained from the model for maximum surge is relatively accurate as proved by the static and steady state tests. But if these results are compared with experimental results some discrepancies are possible. These discrepancies are due to the differences in physical properties assumed in calculations and the factors mentioned under conclusions.

4.0 Conclusions

From the results it is concluded that this model can be used to simulate the water level oscillations in surge tanks due to sudden changes in flow conditions. A better idea of the waterhammer effect can be obtained by comparing the results of sudden closure with slow closure of the valve. The maximum surge oscillations depend on the Darcy's friction factor and the discharge coefficient of the orifice. Although it is assumed that these coefficients are constants in this program, they are functions of Reynold's number of the flow condition. Therefore the discrepancies will be noticed between the theoretical and experimental results. In practice it is very difficult to determine the friction coefficients as they depend on the quality of the surface texture. During the design stage of surge tanks it is necessary to assume that these coefficients are available from previous experiments and introduce a safety factor to account for inaccuracies in these assumed values.

Since the dynamic effect in the surge tank is neglected in this analysis the friction in surge tank walls is not included in the calculations. The wall friction also contributes to the oscillation of water mass in the surge tank and this is another reason for the possible deviations from experimental results. It is also possible to consider the dynamic effect of the water mass in the surge tank for which case the waterhammer pressures in the tunnel will act as a boundary condition to the system.

5.0 References

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(vi) Proceedings of the 1st International Conference on Pressure Surges - BHRA - Canterbury (1972)

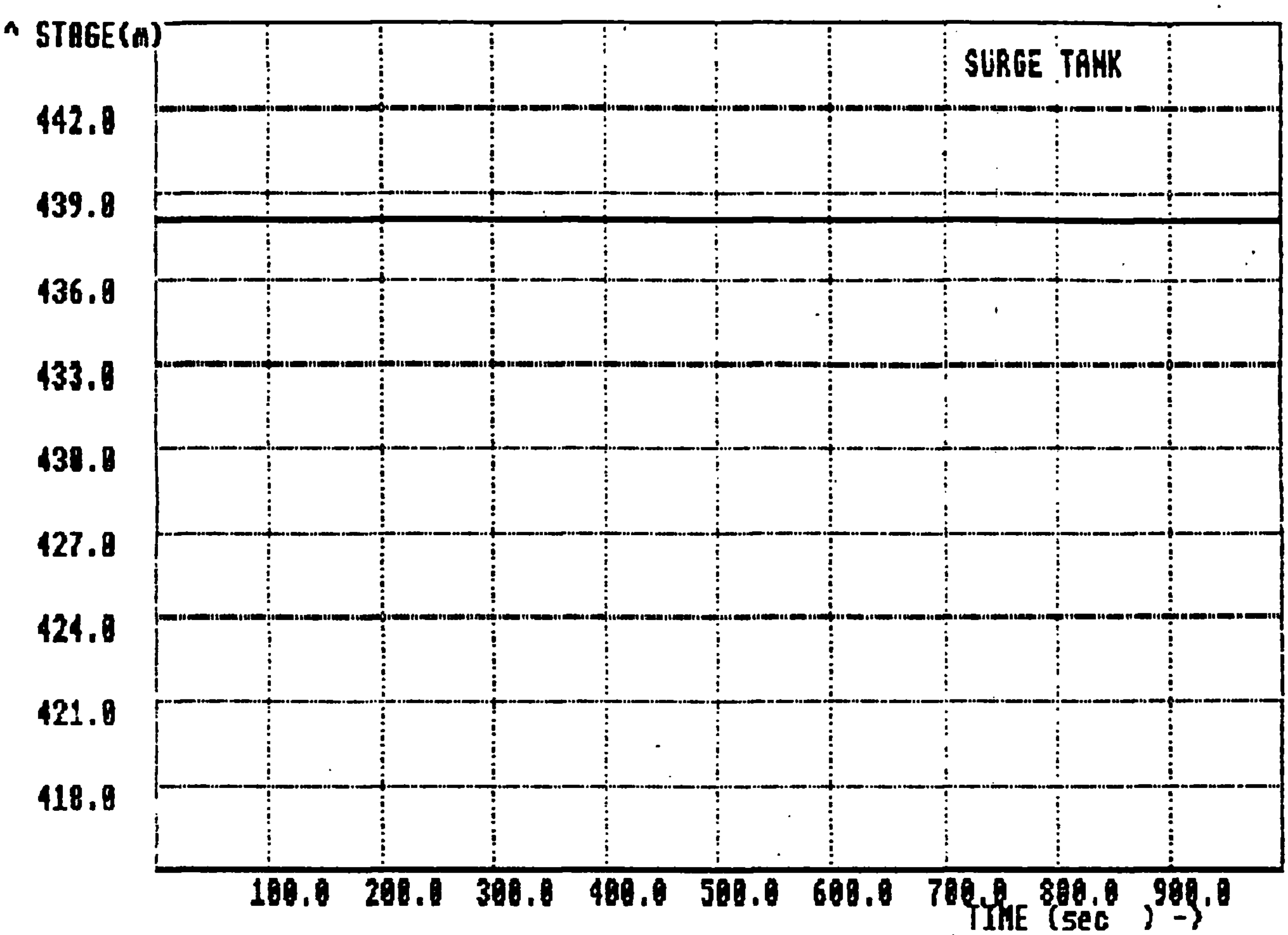


Fig. 6a. Water levels in Surge tank — Static test.

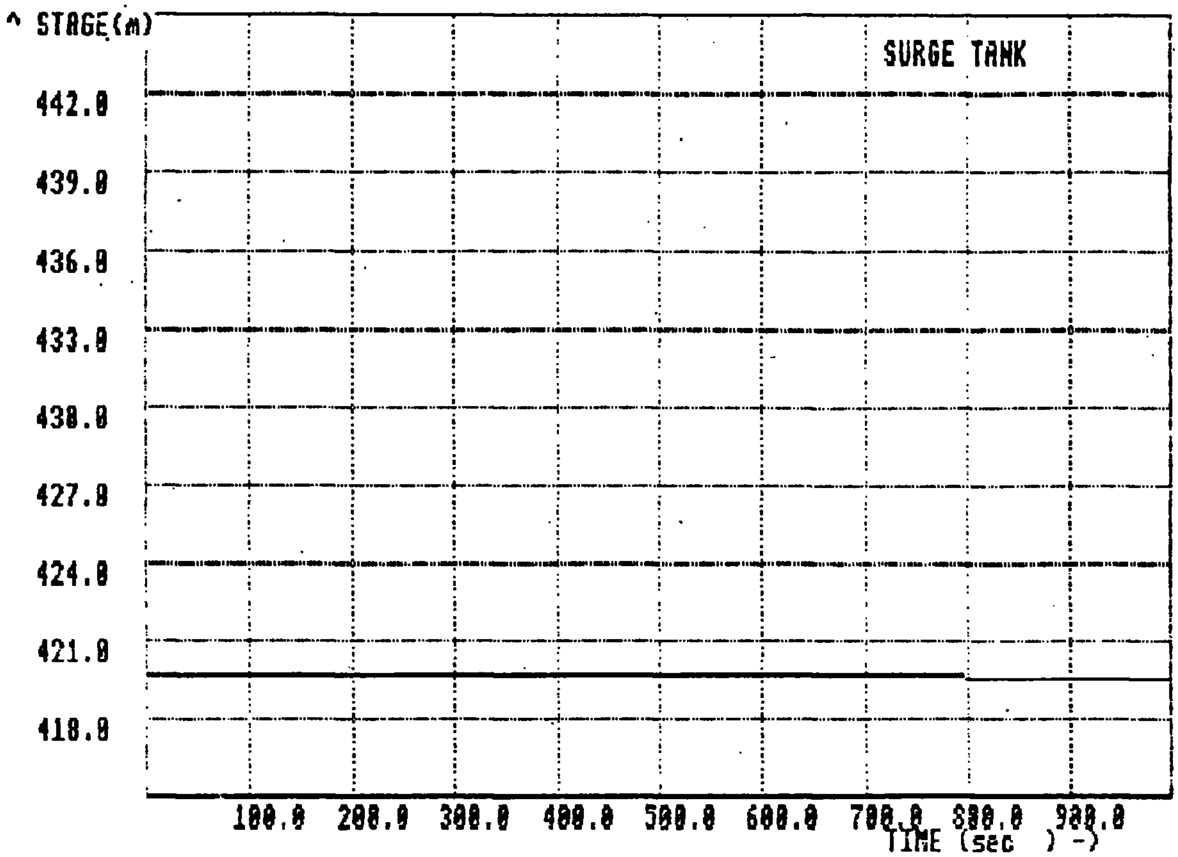


Fig. 6b. Water levels in Surge tank – Steady state test

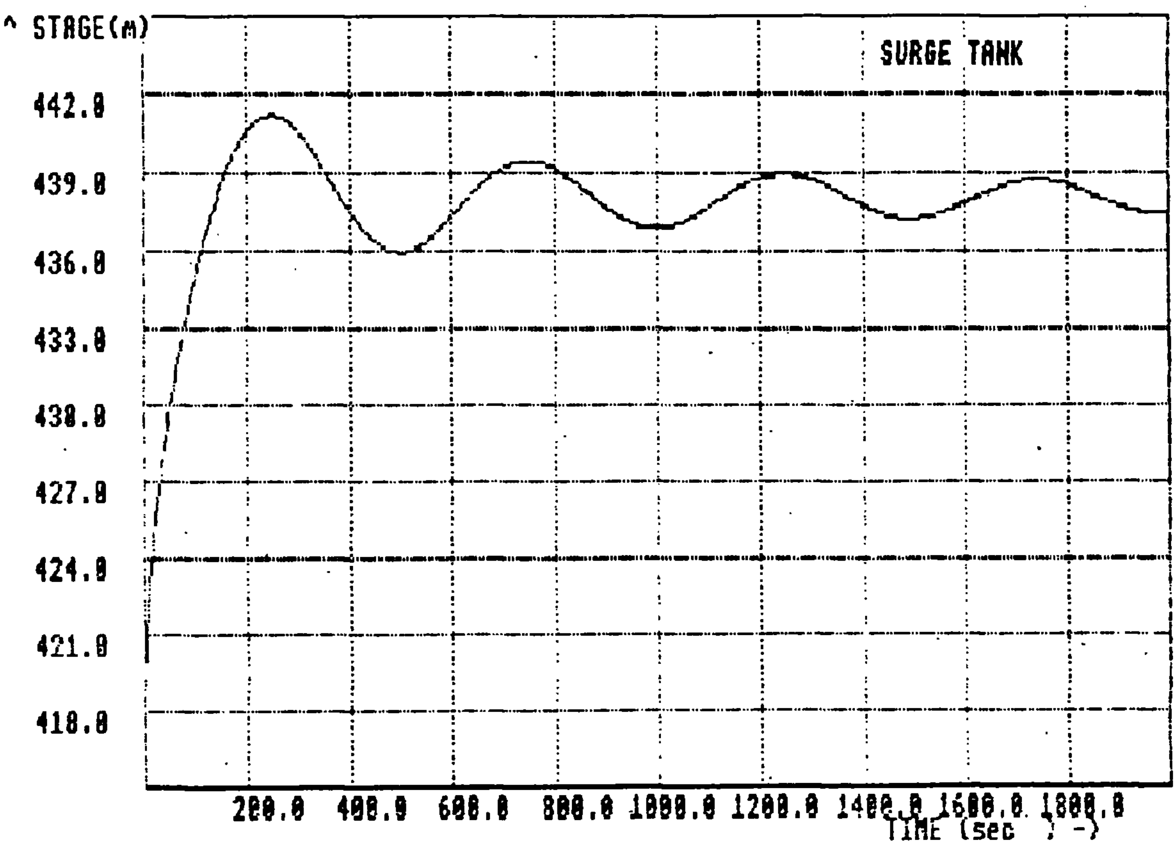


Fig: 6c. Water levels in Surge tank – Sudden closure of valve

(Closing time 5 Sec.)

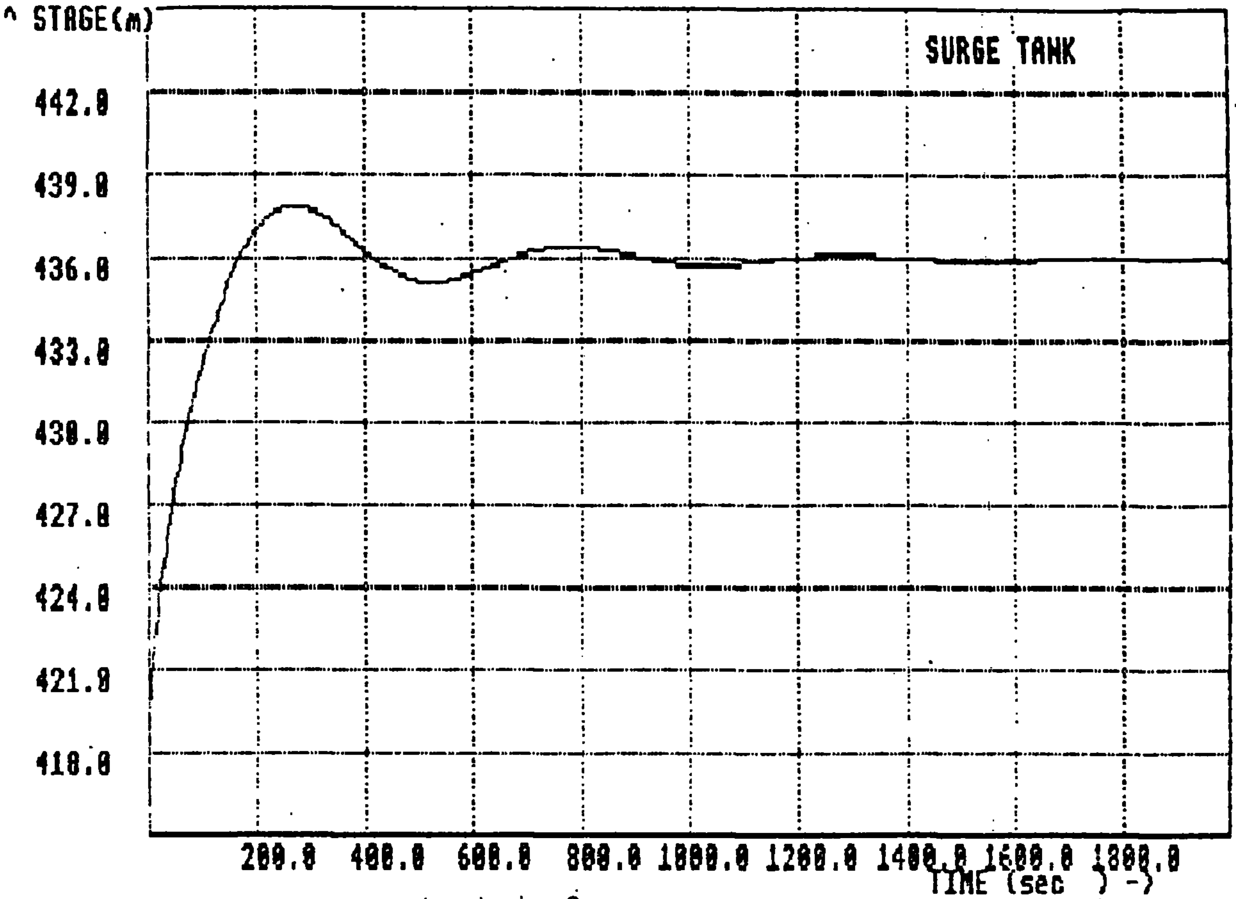


Fig. 6d. Water levels in Surge tank — Partial closure of Valve.
(Closing time 5 Sec.)

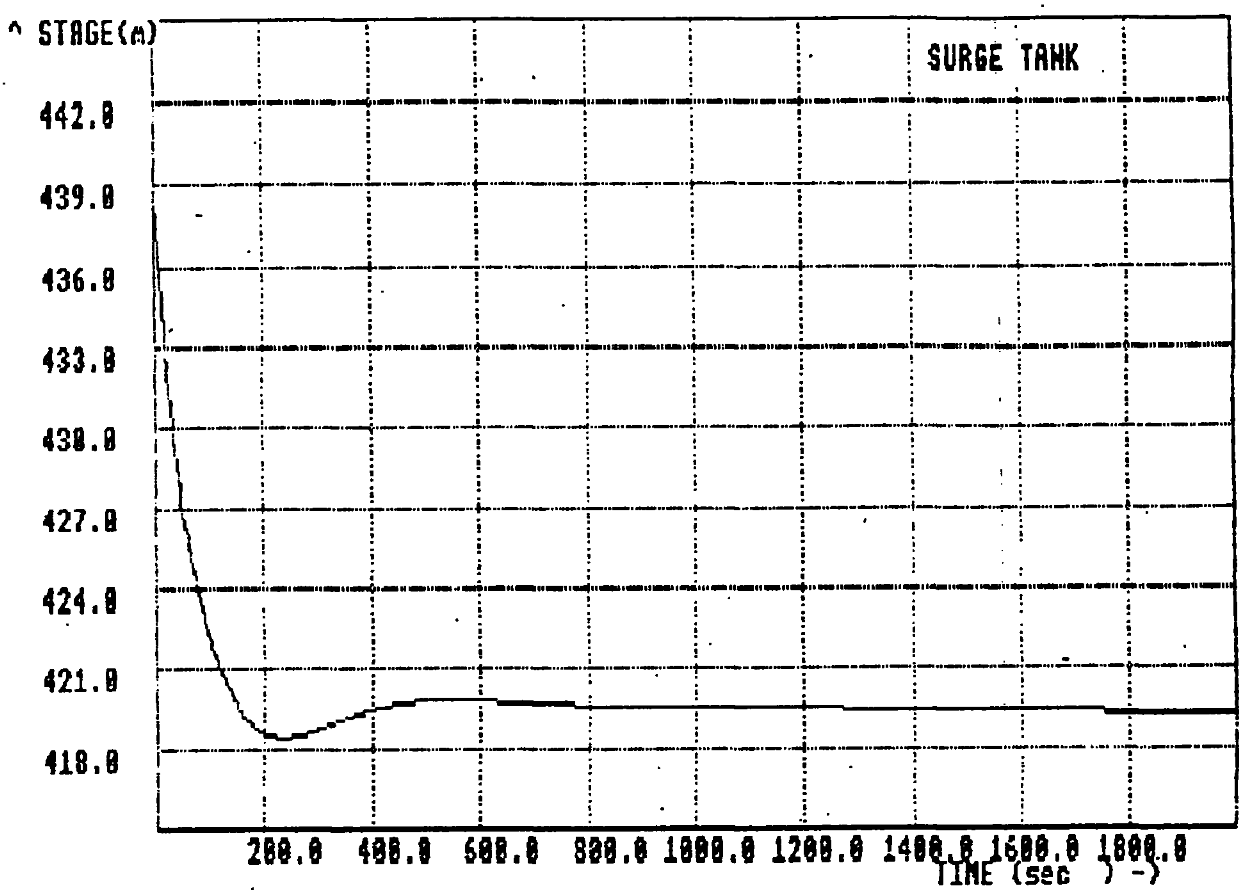


Fig. 6e. Water levels in Surge tank — Sudden opening of valve.
(Opening time 10 Sec.)

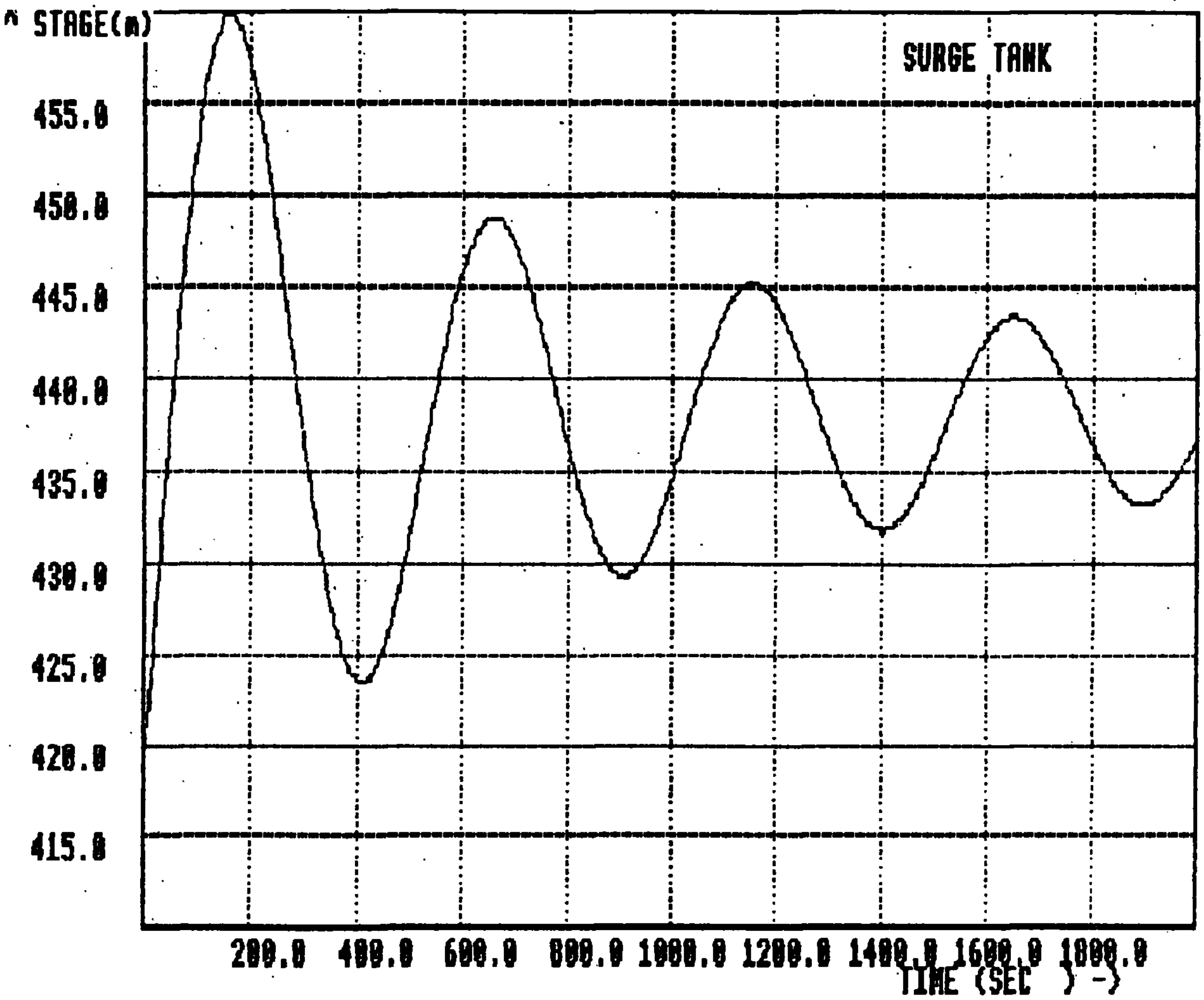
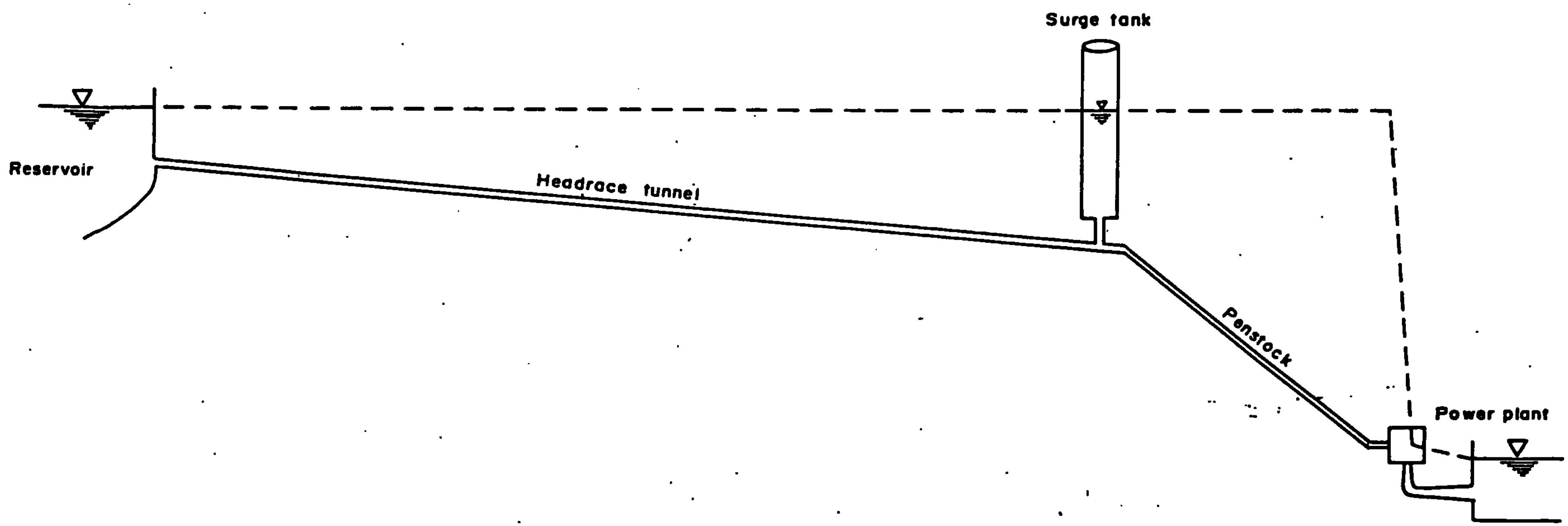


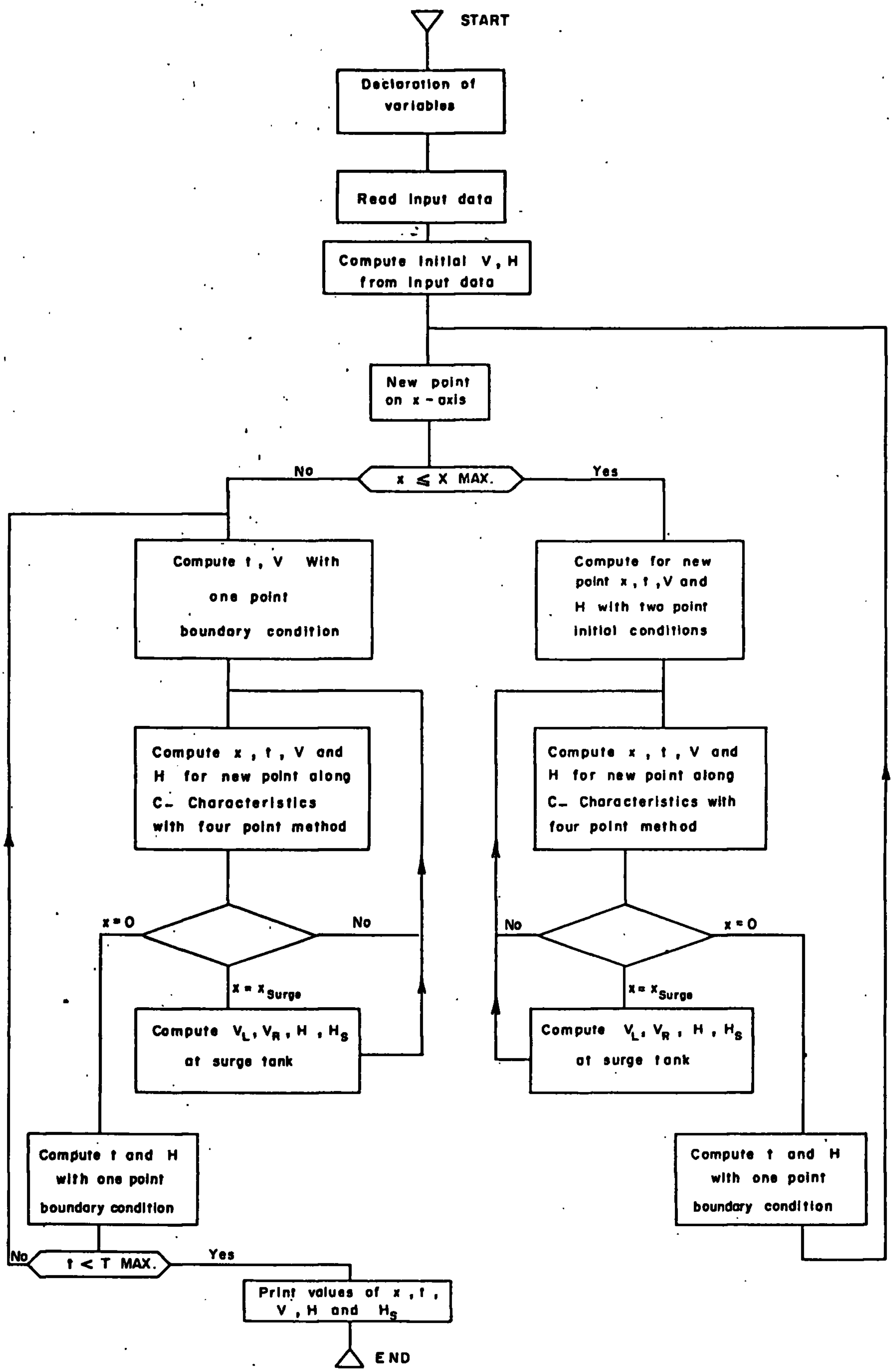
Fig. 6f. Water levels in Surge tank - Sudden closure of valve without an orifice (Closing time 5 Sec.)

Riser-shaft diameter = Tunnel diameter.



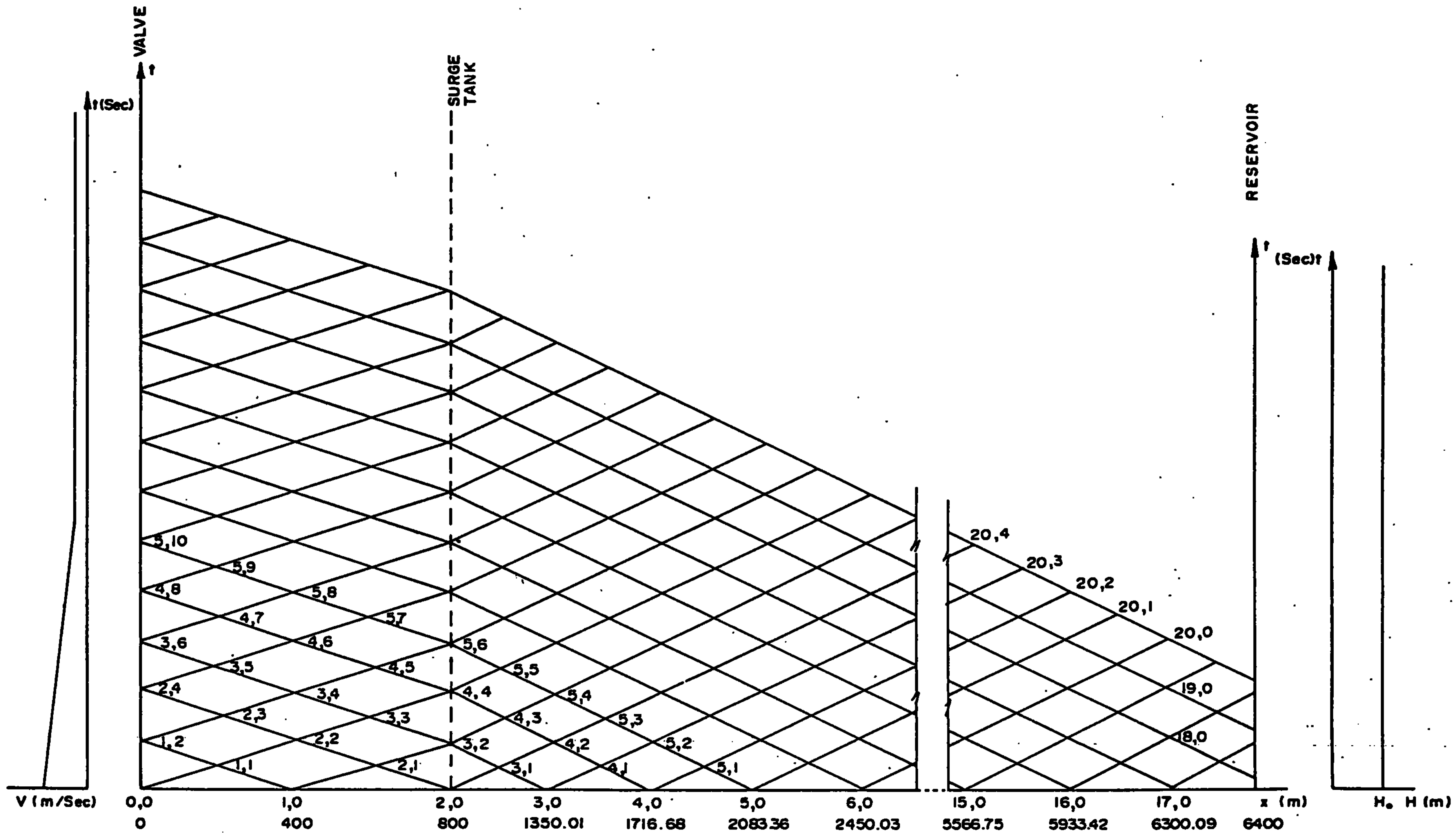
SCHEMATIC DIAGRAM OF A HYDRO-POWER SCHEME

Fig. 7



FLOW CHART OF COMPUTER PROGRAMME.

Fig. 8



TYPICAL CHARACTERISTIC LINES IN x - t PLANE

Fig. 9