NON - LINEAR BEHAVIOUR OF BEAMS

by

S.B.S. Abeykoon

Abstract

A solution procedure based on the finite strip method is presented herein, to analyze the behaviour of rectangular and I beams exhibiting material and geometric non-liearities. The finite strip methods is selected for the analysis as it is easier to extend the proposed method to solve problems involving plates and stiffened plates. Numerical investigations are carried out by applying the proposed method to analyze rectangular and I beams with different boundary conditions.

1.0 Introduction

Different types of beams have been used as structural elements for centuries. It is very difficult, if not impossible, to formulate analytical solutions in closed form for non-linear problems of beams. Numerical solutions, however, can be found with the help of present day computer facilities and with the recent advances in solution procedures.

The well known numerical tool for structural analysis is the finite element method. However, as stated by Y. K. Cheung (1) for many structures having regular geometric plans and simple boundary conditions, a full finite element analysis is unnecessary. The cost of solution can be very high and the machine limitations force the user to be satisfied with a less accurate solution by using lower order elements. These problems could be overcom, especially in analyzing structural elements consisting of beam and plate systems, by the use of finite strip method. In this paper, some application of the finite strip method in the area of non- linear (material and geometric) beam problems will be discussed.

2.0 Theory

2.1 Finite Strip Discretization

The theory follows the familiar finite strip procedure for plate problems [1] A trpical finite strip division of an unstiffened plate is shown in Fig. 1(a). The global rectangular coordinate system xyz coincides with the mid-surface of the plate. Boundary conditions at x=o and x=a are known 'a priori' and will be considered in the choice of displacement functions in the x direction. Any two finite strips are connected along a nodal line, and in the assembly process, the nodal variables are matched along such lines of compatibility. A constant thickness and width are assumed in this formulation thought it is not a necessity.

The present analysis includes both bending and membrane effects, and therefore both in and out of-plane displacements are included as nodal variables. An isolated strip with the local coordinate system is shown in Fig. 1 (b). Displacement nodes are placed at the middle of each side, marked 1 and 2 in Fig. 1(b), with three displacements and one rotation taken as the degrees of freedom, where u and v are the in-plane displacements, w is the out of plane displacement and $\theta=\partial w/\partial y$ is the rotation about the x axis.

2.2 Displacement Functions

The Displacement functions in the x direction, i.e. the longitudinal direction, are chosen so as to satisfy specific boundary conditions at the ends of the strip. Examples of these are given below. The variation of the displacements in they direction, i.e. across the strip, is taken as linear for u and v, and cubic for w. These assumptions ensure slope and displacement compatibility between adjacent strips.

The u, v, and w displacement of a single strip can be written in terms of the nodal displacements by combining the shape functions. The

Dr. S.B.S. Abeykoon Senior Lecturer in Civil Engineering Faculty of Engineering, University of Peradeniya.

expressions for a strip of size a x b are given by,

u= [(1-) u1m] + u2m } gmu() and (1)

$$u = [(1-\eta)u_{1m}+\eta u_{2m}]g_m^u(\xi)$$

$$v = [(1-\eta)v_{1n}+\eta v_{2n}]g_n^v(\xi) \quad \text{and} \tag{1}$$

$$w = [(1-3\eta^2+2\eta^3)w_{1p}+(\eta-2\eta^2+\eta^3)b\theta_{1p}+(3\eta^2-2\eta^3)w_{2p}+(\eta^3-\eta^2)b\theta_{2p}]g_p^w(\xi)$$

where, u_{1m}, u_{2m} etc. are the nodal variables, $\xi = x/a$, $\eta = y/b$ and the summation convention is used for repeated indices, m, n and p. Along the length, the displacements very according to the g (ξ) functions. These functions are given below for the most common displacement boundary conditions.

for structures without stiffeners (eg. rectangular beams)

(a) Simply suported ends

$$g_m^u(\xi) = \begin{cases} \sin m\pi\xi; & m=2,4,6,\ldots \ (constrained) \\ \cos m\pi\xi; & m=1,3,5,\ldots \ (free) \end{cases}$$

$$g_n^v(\xi) = \begin{cases} \sin n\pi\xi; & n=1,3,5,\ldots \ (constrained) \\ \cos n\pi\xi; & n=2,4,6,\ldots \ (free) \end{cases} \tag{2a}$$

$$g_p^w(\xi) = \sin p\pi\xi; \quad p=1,3,5,\ldots$$

(b) Clamped ends

$$g_m^u(\xi) = \begin{cases} \sin m\pi\xi; & m=2,4,6,\ldots \ (constrained) \\ \cos m\pi\xi; & m=1,3,5,\ldots \ (free) \end{cases}$$

$$g_n^v(\xi) = \begin{cases} \sin n\pi\xi; & n=1,3,5,\ldots \ (constrained) \\ \cos n\pi\xi; & n=2,4,6,\ldots \ (free) \end{cases}$$

$$g_p^w(\xi) = \phi_p(\xi) = [\alpha_p(\sinh\beta_p\xi - \sin\beta_p\xi) + (\cosh\beta_p\xi - \cos\beta_p\xi)]/A_p; \quad p=1,3,5,\ldots$$

where

$$A_p = \alpha_p(\sinh 0.5\beta_p - \sin 0.5\beta_p) + (\cosh 0.5\beta_p - \cos 0.5\beta_p) \tag{2b}$$

$$\alpha_p = \frac{\cos\beta_p - \cosh\beta_p}{\sinh\beta_p - \sin\beta_p}$$

and β_p are the solutions of the transcendental equation, $\cosh \beta_p = \sec \beta_p$. These ϕ (ξ) functions are the clamped beam vibration modes.

For structures with stiffeners (eg. I beams)

(a) Simply supported ends

$$g_m^u(\xi) = \cos m\pi\xi; \quad m=1,3,5,\ldots$$

$$g_n^v(\xi) = \sin n\pi\xi; \quad n=1,3,5,\ldots \tag{3a}$$

$$g_p^w(\xi) = \sin p\pi\xi; \quad p=1,3,5,\ldots$$

For structures ends

(3a)

(b) Clamped ends

(3b)

$$g_m^u(\xi) = \frac{\partial\phi_m(\xi)}{\partial x}; \quad m=1,3,5,\ldots$$

$$g_n^v = \phi_n(\xi); \quad n=1,3,5,\ldots \tag{3b}$$

$$g_p^w = \phi_p(\xi); \quad p=1,3,5,\ldots$$

Note that the g_m^u (ξ) functions above are equal to the first derivatives of the corresponding g_p^w (ξ) functions. This choice is necessary to satisfy the plan section constraint of linear beam theory for these built-up sections.

2.3 Strain - Displa Cement Relations

The well known large deflection strain displacement relations for plate bending are considered. These are,

$$\varepsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\varepsilon_y = \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{4}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

where, ε_{x}, ε_{y}, ε_{xy} are the components of strain in two dimensions, and y_{xy} is the engineering shear strain.

When finite strips are used as stiffeners (Fig. 2) the v displacement of the stiffener may be of the same magnitude as the w displacement of the plate. For these strips, midsurface stretching should include quadratic terms in the v displacement. Accordingly, equations (4) are modified for stiffeners by adding corresponding square terms involving the v displacement.

2.4 Constitutive Relations

The material is assumed to follow a bi-linear elastic-plastic law with elastic and plastic modulii E and E t respectively. The von Mises yield criteria and associated hardening flow rule are used. This lead to an incrementally linear relation between stresses ad strains given by (2),

$$\{d\sigma\} = [D_T]\{d\varepsilon\}$$

where{d } and {d} are the incremental changes in the stresses and strains respectively, and $[D_T]$ represents the elastic-plastic constitutive matrix [2]

Stiffness Formulation

The stiffness formulation for the present case follows the standard one for finite elements and only the most pertinent points are given in the following. The displacement functions of equation (1) are written in shape function form as,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N]\{\delta_e\} \tag{6}$$

where [N] is the matrix of shape functions and $(\hat{\delta}_e)$ is the vector of nodal variables. Substitution of equation (6) into the strain displacement relations of equation (5) gives the strains as functions of the nodal variables which can be written symbolically as,

$$\{\varepsilon\} = [B]\{\delta_e\} + \{C_o(\delta_e)\} \tag{7}$$

where [B] is the usual strain displacement matrix, and $\{C_0\}$ contains the quadratic non-linear terms.

The equilibrium equations are obtained from the Virtual work principle. From equation (7), the virtual strains are related to the virtual displacements by,

$$\{\tilde{\varepsilon}\} = [[B] + [C]]\{\tilde{\delta}_e\} \tag{8}$$

where $(\tilde{\varepsilon})$ is the virtual strain vector and $\{\hat{\delta}_e\}$ is the virtual displacement vector. The [C] matrix is linear in$\{\hat{\delta}_e\}$. The virtual work equation can now be written for a single strip as,

$$\int_V \{\tilde{\varepsilon}\}^T\{\sigma\}\,dV = \{\tilde{\delta}_e\}^T\{p\} \tag{9}$$

where{p} is the consistent load vector calculated from the shape functions and the integration is over the volume of the strip. Substitution of equation (8) into equation (9) results in the equilibrium equation for a single strip as,

$$\int_V [[B] + [\tilde{C}]]^T\{\sigma\}\,dV = \{p\} \tag{10}$$

The integral in equation (10) has to be evaluated numerically. However, the stresses are non-linear function of the displacements and hence the Newton-Raphson iteration method was used as the solving technique. Numerical integration was carried out by the use of Gauss quadrature.

3.0 Applications

Several examples have been solved to test the accuracy of the proposed finite strip procedure. In most of the examples discussed in the following, 11 load increments were employed to apply the desired full load. Initially, two load increments, each equal to 5% of the full load were used to avoid any staring problems that might occur. These were followed by nine increments with each increment representing 10% of the full load. In almost all the analyses, a relative accuracy of 0.001 for each displacement variable was stipulated and convergence was achieved in less than six iterations.

3.1 Simply Supported Rectangular Beam -Uniform Load

The deflected shape and the bending moment distribution predicted by the proposed scheme with a single mode, linear elastic analysis are presented in Figures 3 and 4. Central deflection response for a one mode, large deflection analysis with and without material nonlinearities are presented in Figure 5. The numbers (2,1,1) shown in the figure represent the (m,n,p) mode numbers. The elastic solution agrees very well with the Timoshenko solution (3) . The elastic-plastic solution is compared to two finite element solution (4,5) in the figure and very good agreement is seen with the results of Soreide at al.(4).

3.2 Clamped Rectangular Beam - Uniform Load

The predicted deflected shape and the bending moment distribution with a one mode linear elastic analysis are presented in Figure 6 and 7. Central deflection results for this case are shown in Figure 8 and indicate an important feature in the selection of modes for large displacements. For the elastic case it is seen that the finite strip results are highly dependent on the mode shapes used for the u displacement. As a general rule, if the w displacement is represented by a mode shape given by $\cos p\pi\xi$, then u should have a shape of $\sin 2p\pi\xi$.This is a result of the equilibrium requirement of a uniform axial force. Since the first mode shape for w in this case is given approximately by $(1-\cos 2\pi\xi)$, it follows that u should be given by $\sin 4\pi\xi$, which produces aresult in much closer agreement to the Timorous solution [3] than does $u = \partial\phi_1/\partial x$ or $u=\sin 2\pi\xi$.The $\sin 4\pi\xi$ shape used in this example has been verified by comparing with the u displacement variation obtained by a finite element computer program FENTAB [6]. In Figure 8 the plastic large deflection results are also compared with those from FENTAB, and show reasonable agreement.

3.3Simply Supported I Beam - Uniform Load

Elastic large deflection results of a simply supported I beam are presented in Figure 9, along with the section dimensions and material properties. Finite strip results were obtained by employing six finite strip, two each in the top flange, bottom flange and the web. In this case two modes are needed to represent the u displacement, even though only one w mode is used. One u mode accounts fro the bending strains, and so should have the shape given by $\partial w / \partial x$, or cos $2\pi\xi$in this case, while the other u mode accounts for the axial forces in the large displacement region of the response, as discussed in the previous section. Central deflections are in very good agreement with finite element results obtained by FENTAB [6].

The comparison of predicted central deflections with FENTAB for an elastic-perfectly plastic material, including large deflections, is presented in Figure 10, where it is seen that the agreement is excellent is excellent. When the central deflation approaches twice the beam depth, the FENTAB solution runs asymptotic to the plastic string limit. The finite strip solution, on the other hand, agrees with the solution obtained using a one mode Galerkin procedure on the governing differential equation for a plastic string. This is expected, as the assumed displacement variation in the Galerkin analysis is the same as that in the finite stip, namely sin $\pi\xi$

Axially restraining an I beam at different locations along the depth makes a considerable difference to the load-deflection response. Results for central deflection of the same I beam in an elastic large deflection analysis are presented in Figure 11 for the different support points A,B,C,D and E. These results were obtained by modeling the web with four equal width finite strips. The support point was moved from A through B by changing the boundary conditions at the appropriate nodal line. In a liner elastic analysis, points A and E should yield the same displacement pattern as they are equidistant from the centroidal axis. Therefore, near the origin, both these curves have the same slope. This is also true for the points B and D However, as the axial; straining starts to take place, point A produces the stiffest solution and point E produces the most flexible solution as one might expect.

3.4 Clamped I Beam - Uniform Load

Elastic non-linear results results for this case are compared with FENTAB results in Figure 12, and very good agreement in seen when two u mode are used, namely $\partial \phi / \partial \xi$ and sin $4\pi\xi$. Results for the non-linear elastic-perfectly plastic analysis are presented in Figure 13. The finite strip results are in considerable error in the intermediated range of deflections when compared with FENTAB. This arises because the one mode use for the w displacement does not allow for the hinges in a bending collapse mechanism to occur, even though full plastification may occur at any section. For large deflections when the beam is acting as a plastic string, agreement with FENTAB is much better. This poor performance could perhaps be overcome with the addition of a w mode that essentially had very large curvatures at the ends and centre only. This aspect needs further investigation.

4.0 Conclusions

The proposed finite strip procedure gives very good results for non-linear problems involving rectangular and I beams even when working with only one displacement mode in the longitudinal direction. Method is already extended for similar problems involving plates and stiffened plates. It is believed that any extensions towards problems with non-uniform loads will not be difficult but will definitely be expensive in termsof computer time. However, finite strip solution procedure will be much cheaper compared to finite element techniques. The proposed method can also be extended to analyze dynamic problems.

5.0 Acknowledgment

The author is indebted to Professors M.D. Olson and D.L. Anderson of the University of British Columbia, Vancouver, Canada, for their guidance and encouragement. This research has been supported by the Canadian Natural Sciences and Engineering Research Council andCanadian Defense through grants and contracts respectively.

6.0 References

1. Y.K. Cheung, Finite strip Method in Structural Analysis, Pergamon Press, Oxford, 1976.

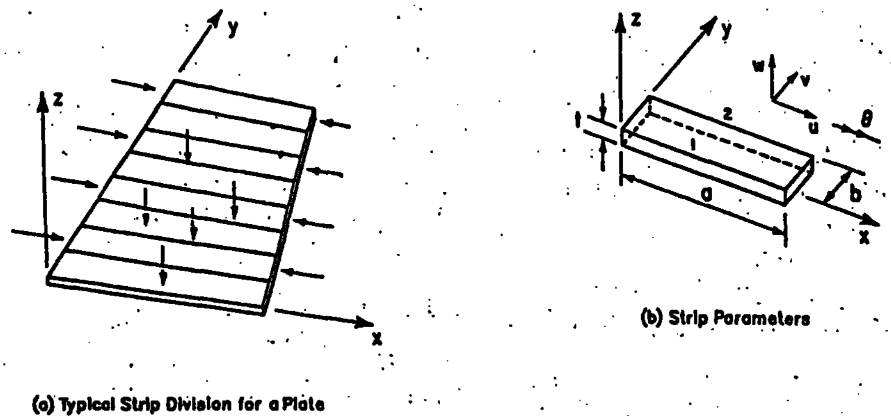
2. O.C. Zienkiewicz, The Finite Element Method, 3rd Edition, McGraw-Hill, London, 1973.

3. S.P. Timoshenko and S. Woinowsky - Kreger, Theory of Plates and Shells, 2nd Edition, London, 1953.

4. T.A. Soriede, T. Moan and N.T. Nordsve, on the Behaviour and Design of Stiffened plates in Ultimate Limit State, Journal of Ship Research, 22,238-244, 1978.

5. J. Backlund, Finite Element Analysis of Non-Linear Structures, PhD thesis, Chalmers Iekniska Hogskola, Goteberg, Sweden, 1973.

6. B. R. Folz, Numerical Simulation of the Non-Linear Transient Response of Slender Beams, MASc thesis, U.B.C., Vancouver, Canada, 1986.



Finite strip modelling of a plate
Figure 1

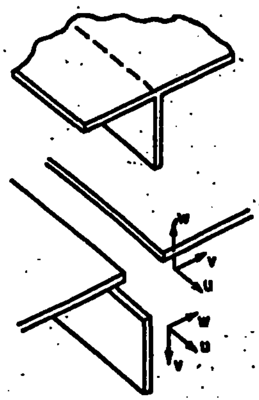
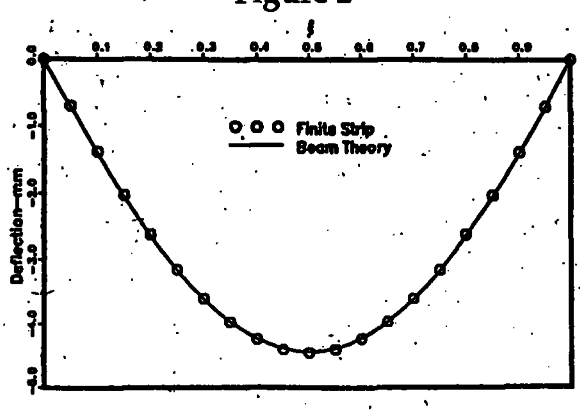
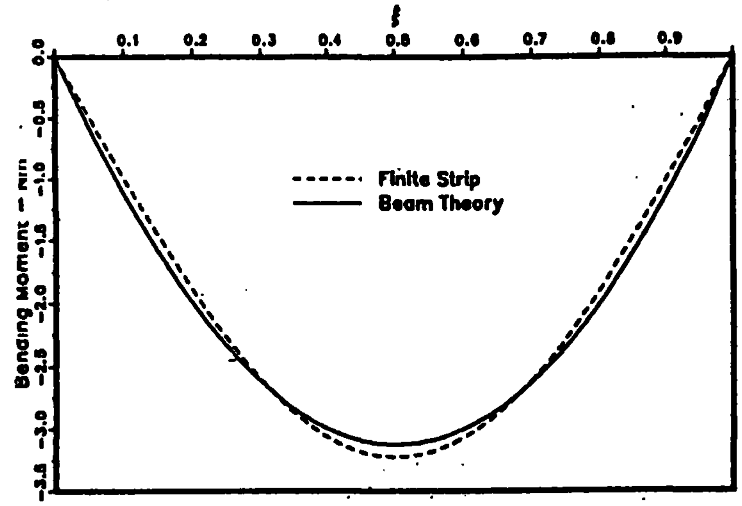


Plate - stiffener assemblage
Figure 2 -



Deflected shape of the simpply supported rectagular beam
Figure 3 -



Bending moment distribution of the simply supported rectangulsar beam
Figure 4 -

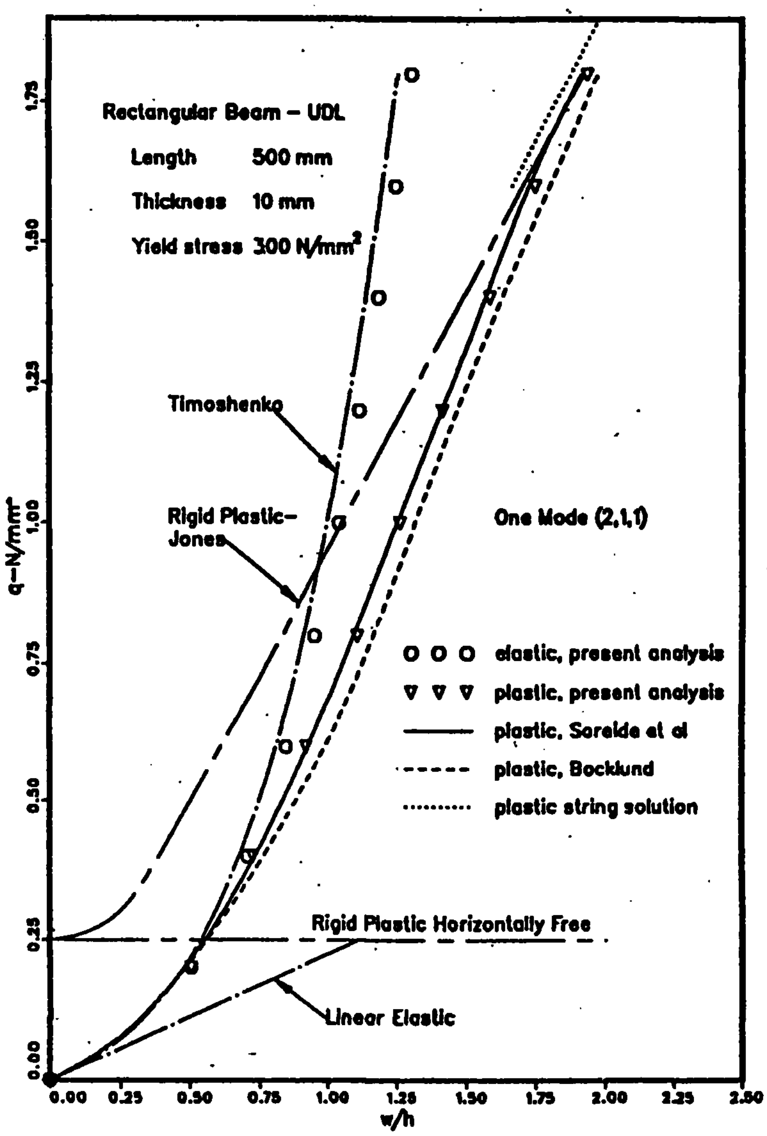
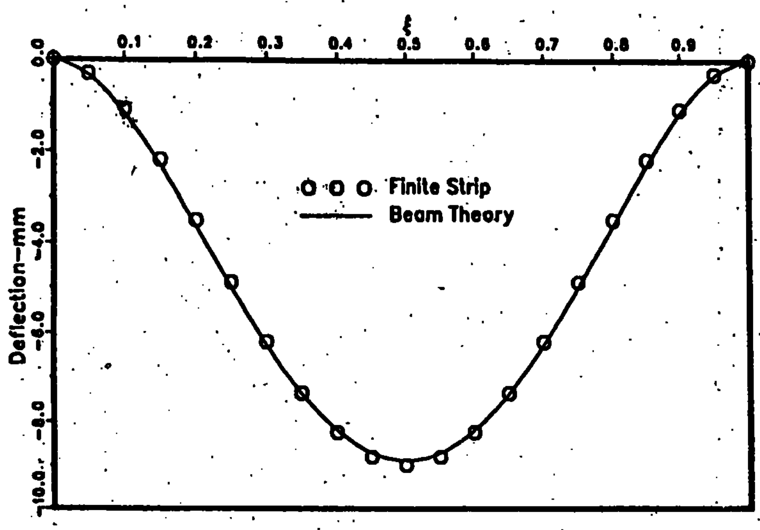


Figure 5 - Central deflections of the simply supported rectangular beam



Deflected shape of the clamped rectangular beam
Figure 6 -

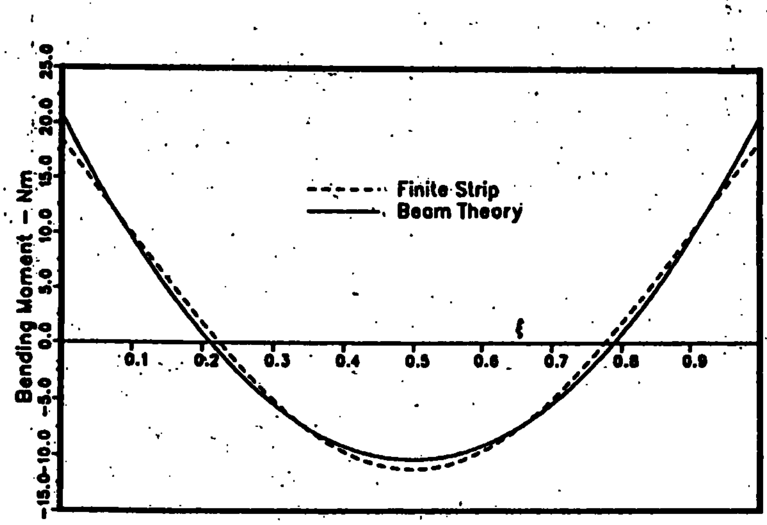


Figure 7 - Bending moment distribution of the clamped rectangular beam

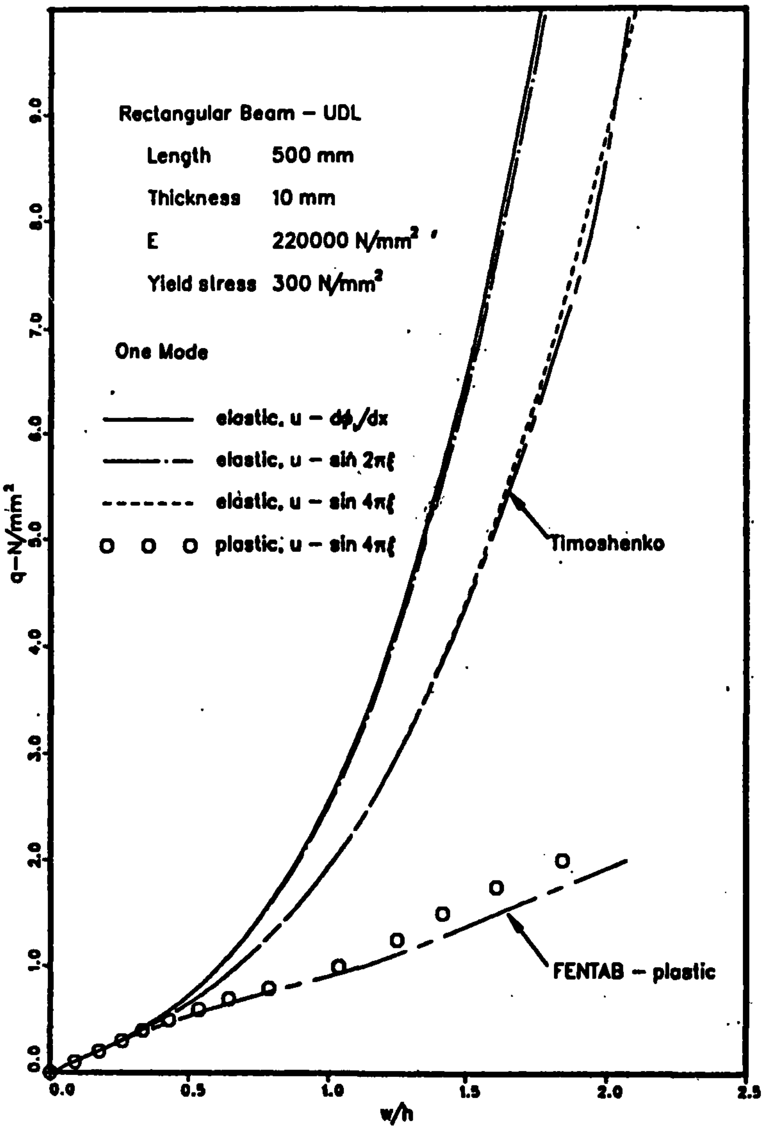


Figure 8 - Central deflections of the clamped rectangular beam

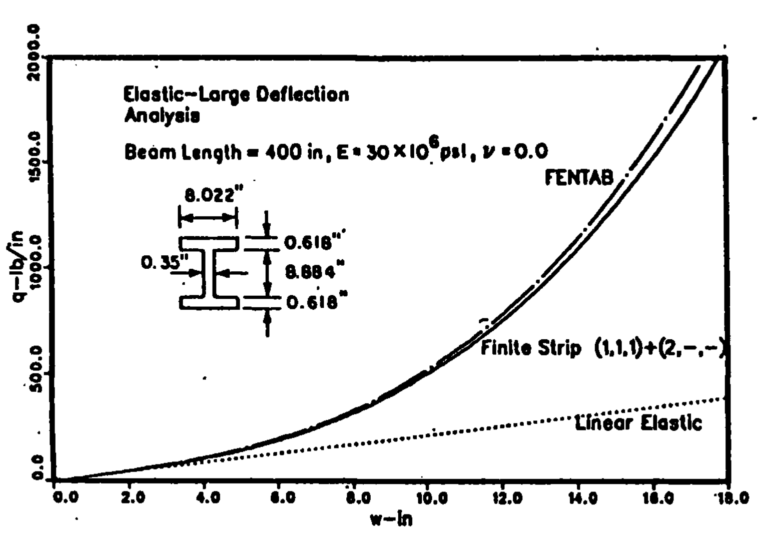


Figure 9 - Central deflections of the simply supported I beam

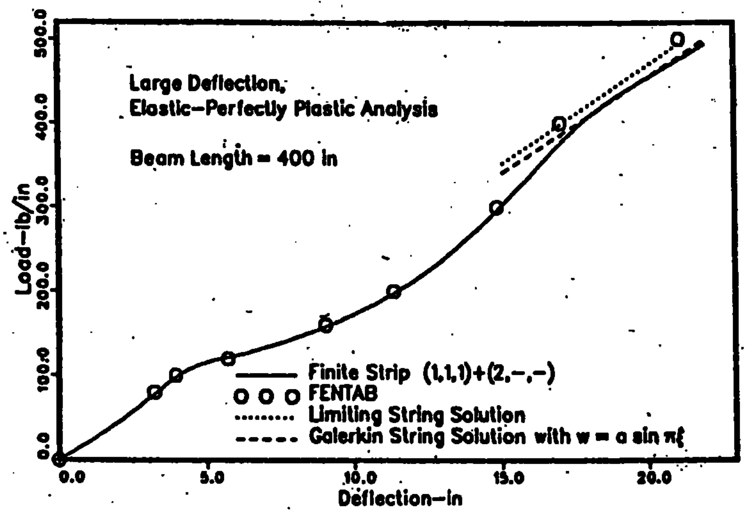


Figure 10 - Central deflections in an elastic - perfectly plastic analysis

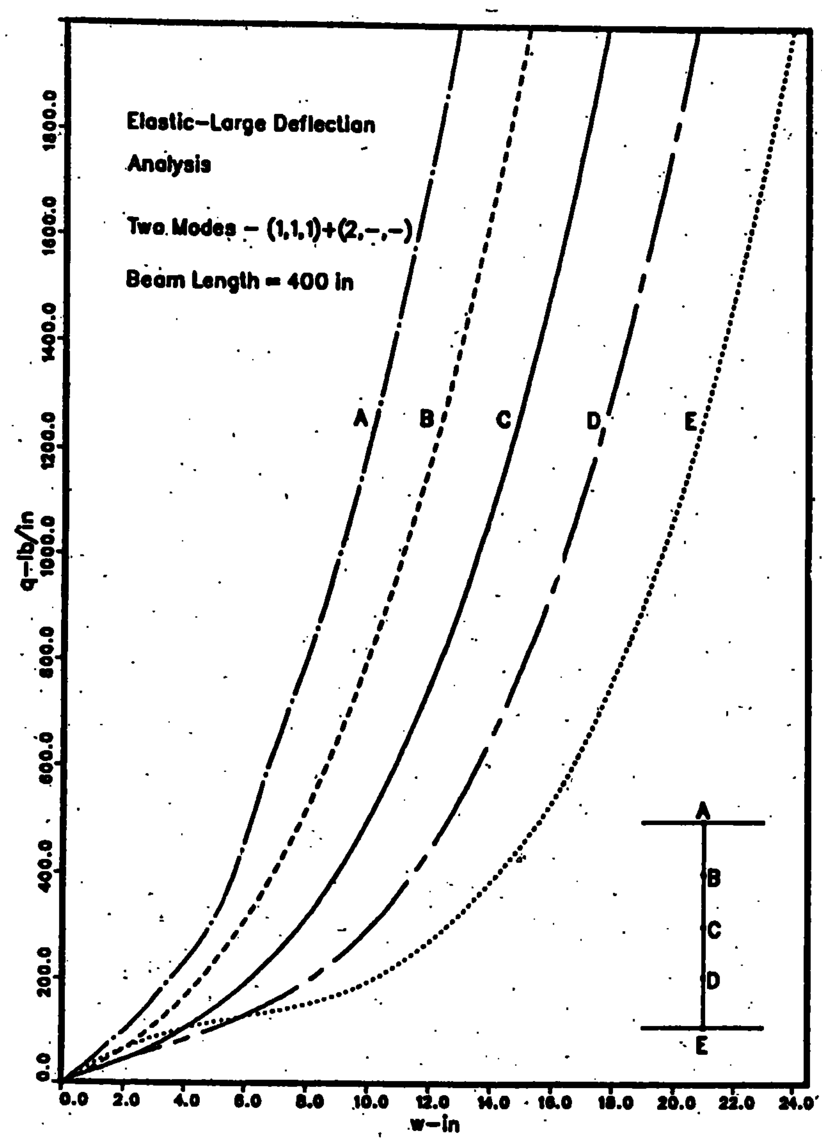


Figure 11- Central deflections with varying support points

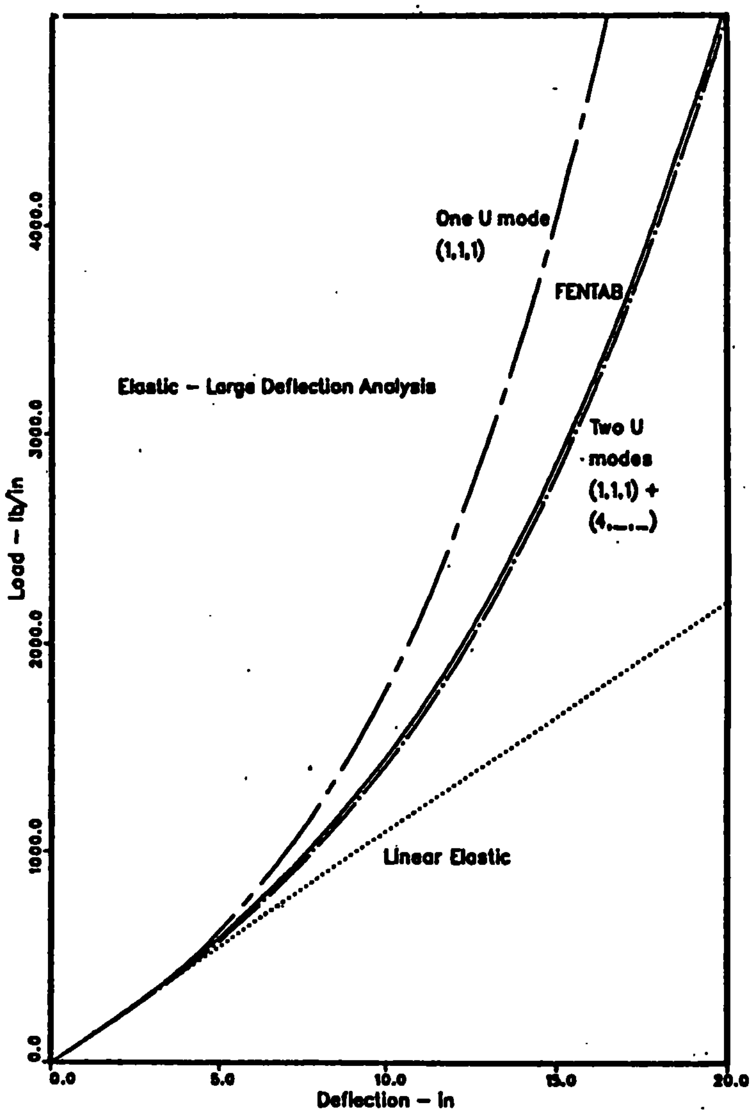


Figure 12 - Central deflections of the clamped I beam

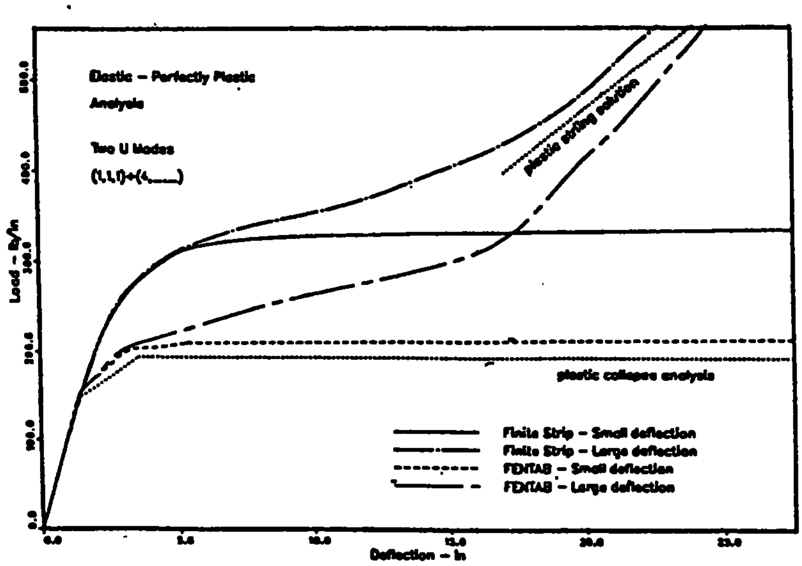


Figure 13 - Elastic-perfectly plastic analysis of the clamped I beam