A STUDY ON THE PARAMETERS OF THE MUSKINGUM CUNGE MODEL FOR FLOOD ROUTING

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Abstract

The Muskingum method is presented as a finite difference approximation of the parabolic differential equation. Therefore, a relationship which enables to find an approximate explicit expression for the time related weighting coefficient is obtained. As this relationship also relates to the physical river characteristics, it becomes more meaningful when choosing a value for. The stability of the Muskingum-Cunge parameters are investigated and relationships of discharge to parameter values are established in order to make it easier to decide on these values for a given situation.

1.0 Introduction

The Muskingum - Cunge method which is used practically to route flood waves, or simply calculate the discharge in a river is a convenient method as the computer calculations can be carried out even with a computer of limited capacity. However, as the Muskingum formula is directly derived from a finite different scheme replacing the partial differentials in the original equation, this scheme introduces an error into the analytical solution of the original equation (1). This error takes the form of an artificial attenuation of the rate of flow as well as modifies the velocity of propagation (celerity) of the resulting wave (2).

In order to obtain accurate results, is is essential that we keep this error to a minimum, This is only possible by taking accurate values for the time & space related weighting parameters (3). To obtain accurate values for these parameters the relevant changes & relationships these parameters hold with respect to change in discharge make an attempt to establish these relationships so that we may be able to decide on more accurate values for these parameters. By expressing the Muskingum - Cunge equation in the finite difference approximation format it becomes easier to understand the strengths and weaknesses of this equation.

2.0 Muskingum Fromula And The Finite Diffrence Scheme

The Muskingum formula is a finite difference approximation of another differential equation derived from the Saint- Venant equation neglecting the inertia terms. Saint - Venant equations, which are for gradually varying transient flow, are as follows.

Continuity equation

$$\frac{\partial h}{\partial t} + \frac{1}{b}\frac{\partial Q}{\partial x} = 0 \qquad (1)$$

dynamic equation

$$\frac{1}{g}\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = i_o - \frac{\partial h}{\partial x} - \phi Q|Q| \qquad (2)$$

Where io is the bed slope, h is the depth of water. Q is the discharge, b is the width, v is the mean velocity and ø is the resistance coefficient. A is the cross sectional area.

As in general A = A(h), the following equation is obtained.

$$\frac{\partial h}{\partial t} = \frac{dh}{dA}\frac{\partial A}{\partial t} = \frac{1}{b}\frac{\partial A}{\partial t} \qquad (3)$$

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For any given point along the river, assuming $A = A(Q)$, will give eq. (4)

$$\frac{\partial h}{\partial t} = \frac{1}{b}\frac{dA}{dQ}\frac{\partial Q}{\partial t} \qquad (4)$$

Substituting this into the continuity eq. (1) will result in eq. (5)

$$\frac{dA}{dQ}\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad (5)$$

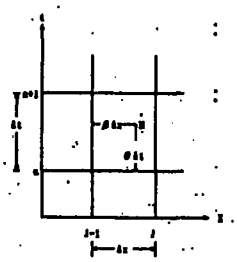
This is a first order, partial differential equation, which is non linear in so far as dA/dQ itself is a function of discharge. Denoting celerity as C and taking the differential oprator as D, eq. (5) could be rewritten as eq. (6)

$$DQ = \frac{1}{C}\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} = 0 \qquad C = \frac{dQ}{dA} \qquad (6)$$

Assuming a difference scheme in the (x,t) plane as in Fig-1, an implicit finite difference approach to numerically solve eq. (6) could be attempted.

$$\frac{\partial Q}{\partial t} \approx \frac{\theta\,(Q_{j-1}^{n+1} - Q_{j-1}^{n}) + (1-\theta)\,(Q_j^{n+1} - Q_j^{n})}{\Delta t} \qquad (7)$$

$$\frac{\partial Q}{\partial x} \approx \frac{\beta\,(Q_j^{n+1} - Q_{j-1}^{n+1}) + (1-\beta)\,(Q_j^{n} - Q_{j-1}^{n})}{\Delta x}$$



: Time related weighting parameter
: Space related weighting parameter

Fig.1 Difference Scheme (x,t) plane

Weighting with ß and ⊖ may be justified if the resulting finite difference equation is an acceptable approximation of differential eq. (6) and if the solution converges with that of eq. (6). Based on the finite difference scheme (7) approximating the differential operator D of eq. (6) as Dh, yields eq. (8)

$$D_hQ = \frac{1}{C}\,\frac{\theta\,(Q_{j-1}^{n+1} - Q_{j-1}^{n}) + (1-\theta)\,(Q_j^{n+1} - Q_j^{n})}{\Delta t} + \frac{\beta\,(Q_j^{n+1} - Q_{j-1}^{n+1}) + (1-\beta)\,(Q_j^{n} - Q_{j-1}^{n})}{\Delta x} = 0 \qquad (8)$$

Assuming the grid functions etc. are expanded as a Taylor series about the point (J-1,n), the approximation error R made by using the difference operator Dh is as in eq.(9).

$$R = \left\{\left(\theta - \frac{1}{2}\right) + \frac{C\Delta t}{\Delta x}\left(\frac{1}{2} - \beta\right)\right\}\frac{\partial^2 Q}{\partial x^2}\cdot\Delta x + \frac{C}{2}\left\{\theta\left(\frac{\Delta t}{\Delta x} - \frac{1}{C}\right) + \frac{\beta\Delta t}{\Delta x}\left(1 - \frac{C\Delta t}{\Delta x}\right) + \frac{C}{3}\frac{\Delta t}{\Delta x} + \frac{2}{3C} - \frac{\Delta t}{\Delta x}\right\}\frac{\partial^3 Q}{\partial x^3}\cdot\Delta x^2 + \cdots \qquad (9)$$

The finite difference operator Dh. based on scheme (7) is also an approximation of a differential operator D1. such that,

$$D_1Q = \frac{1}{C}\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x} + \gamma\frac{\partial^2 Q}{\partial x^2} = 0 \qquad (10)$$

$$\gamma = \left\{\left(\theta - \frac{1}{2}\right) + \frac{C\Delta t}{\Delta x}\left(\frac{1}{2} - \beta\right)\right\}\Delta x \qquad (11)$$

Substituting ß = 1/2 into eq. (8) the following relationship which is the well known Muskingum formula is obtained. =1/2 has been selected as this is the central point for the weighting parameter.

$$Q_j^{n+1} = C_1 Q_{j-1}^{n} + C_2 Q_{j-1}^{n+1} + C_3 Q_j^{n}$$

$$C_1 = \frac{K\theta + \Delta t/2}{K(1-\theta) + \Delta t/2} \qquad C_3 = \frac{K(1-\theta) - \Delta t/2}{K(1-\theta) + \Delta t/2}$$

$$C_2 = \frac{-K\theta + \Delta t/2}{K(1-\theta) + \Delta t/2} \qquad K = \frac{\Delta x}{C} \qquad (12)$$

When calculations are made using the Muskingum formula, it will result in a damping effect. This is due to the Muskingum formula being an approximation of the equation, and damping must result from the introduction of coefficients ϴ and ß. Here it could be seen that the Muskingum method can be considered as a finite difference approximation of the parabolic differential equation eq. (10).

3.0 Analysis of The Parabolic Differential Equation

Neglecting the inertia terms in the dynamic eq. (2). will result in the following equations.

$$\frac{\partial h}{\partial t} + \frac{1}{b}\left(\frac{\partial Q}{\partial x}\right) = 0 \tag{13}$$

$$I_0 - \frac{\partial h}{\partial x} - \phi Q|Q| = 0 \tag{14}$$

Here,

$$\begin{aligned} Q &= Q_0 + Q' \\ h &= h_0 + h' \\ \emptyset &= \emptyset_0 + \emptyset' \end{aligned} \tag{15}$$

Where Q_0, h_0 and $\emptyset_0$ are discharge, depth and resistance coefficients for steady flow and Q/, h/ and ø / are small changes in their quantities.

Substituting eq. (15) into eq. (14).

$$I_0 - \frac{\partial h}{\partial x} - \phi_0 Q_0|Q_0| - 2\phi_0|Q_0|Q' - \phi' Q_0|Q_0| = 0 \tag{16}$$

Differentiating eq. (13) with respect to distance and eq. (16) with respect to time and combining them results in eq (17).

$$\frac{1}{b}\frac{\partial^2 Q}{\partial x^2} - 2\phi_0|Q_0|\frac{\partial Q'}{\partial t} - Q_0|Q_0|\frac{\partial \phi'}{\partial t} = 0 \tag{17}$$

Substituting eq. (13) into this equation gives eq. (18)

$$\frac{-2b\phi_0}{Q_0(d\phi_0/dh)}\frac{\partial Q'}{\partial t} + \frac{\partial Q}{\partial x} + \frac{1}{Q_0|Q_0|(d\phi_0/dh)}\frac{\partial^2 Q}{\partial x^2} = 0 \tag{18}$$

Eq. (18) is a parabolic type differential equation [3]. For uniform flow eq. (14) becomes as follows.

$$I_0 = \phi_0 Q_0|Q_0| \tag{19}$$

For non - uniform steady flow as the variation of Q is small the following equation holds good.

$$Q_0 = (I_0)^{1/2}\phi_0^{-1/2} \tag{20}$$

and comparing eq. (6) and eq. (18) for a wide rectangular channel, will result in eq. (21).

$$C = \frac{dQ}{dA} = -\frac{1}{2b} I_0^{1/2}\phi_0^{-3/2}\cdot\frac{d\phi_0}{dh} = -\frac{Q_0}{2b\phi_0}\frac{d\phi_0}{dh} \tag{21}$$

The condition for the finite difference eq. (10) to be a second order approximation of eq. (18) is as follows.

$$\left\{\left(\theta-\frac{1}{2}\right)+\frac{C\Delta t}{\Delta x}\left(\frac{1}{2}-\beta\right)\right\}\Delta x = \frac{1}{Q_0|Q_0|\cdot d\phi_0/dh} \tag{22}$$

Substituting eq. (21) into eq. (22) and assuming ß = 1/2 since it is the central value, will give eq. (23).

$$\theta = \frac{1}{2}\left(1-\frac{1}{b\phi_0 C|Q_0|\Delta x}\right) \tag{23}$$

Now using eq. (19) will result in eq. (24)

$$\theta = \frac{1}{2}\left(1-\frac{Q_0}{I_0 b C\Delta x}\right) \tag{24}$$

This relationship enables an approximate explicit expression to be found for. ϴ Nevertheless this will not always yield satisfactory results. However asϴ obtained by this relationship is related to the physical river characteristics, even if the calculations for which it is used fail to reproduce the observed flood, this is the only meaningful coefficient to be adjusted as the value of

Only meaningful coefficient to beadjusted as the value of $\Delta X/\Delta t$ has no physical meaning.

4.0 Comparison of the weighted Residual Model Parameters

In this section a comparison between the Muskingum model and the finite difference equation discretized (broken down and integrated) using the weighted residual method is done. The equation from the weighted residual method is as follows. (Refer [4] for derivation of the equation).

$$\frac{\beta\left(Q_j^{n+1}-Q_{j-1}^{n+1}\right)+(1-\beta)\left(Q_j^{n}-Q_{j-1}^{n}\right)}{\Delta x_j}+\frac{(\lambda)_{j-1}}{3\Delta t}\cdot\left\{\left(1+\frac{\alpha}{1+\alpha}\right)\cdot\left(Q_{j-1}^{n+1}-Q_{j-1}^{n}\right)+\left(1+\frac{1}{1+\alpha}\right)\cdot\left(Q_j^{n+1}-Q_j^{n}\right)\right\}=(q)_{j-1} \tag{25}$$

Here it could be seen that the form and coefficient of eq. (25) is similar to the finite difference scheme proposed by Cunge.

Moreover, if.

$$\theta=\frac{1}{3}\left(1+\frac{\alpha}{1+\alpha}\right) \tag{26}$$

Note -

The relationship of θ and α expressed as in eq.(26) is such that eq.(25) and (27) become simillar. However as this relationship has been obtained by just interchanging and equating equations the real physical reason as to why it is so must be investigated.

eq. (25) could be written as eq. (27)

$$\frac{\beta\left(Q_j^{n+1}-Q_{j-1}^{n+1}\right)+(1-\beta)\left(Q_j^{n}-Q_{j-1}^{n}\right)}{\Delta x_j}+\frac{(\lambda)_{j-1}}{\Delta t}\cdot\left\{\theta\left(Q_{j-1}^{n+1}-Q_{j-1}^{n}\right)+(1-\theta)\left(Q_j^{n+1}-Q_j^{n}\right)\right\}=(q)_{j-1} \tag{27}$$

This is same as scheme (7) which introduces a weighting coefficient θ in the finite difference representation of the time derivative and a weighting coefficient ß in the finite difference representation of the space derivative at the point M as in Fig 1.

Introducing thetravel parameter $K = \Delta x_j / \langle C \rangle_j^{j-1} = \Delta x_j \langle \lambda \rangle_j^{j-1}$ and ß = 1/2, the solution of eq. (27) for Q_j^{n+1} can be reduced to a form identical to the classical Muskingum - Cunge equation, eq. (12)

The weighting factor α to be meaningful it must be within the range zero to one. As α is related to θ as of eq. (26), θ will have to take a reference range corresponding to the reference range of . Therefore, θ will have to vary between 1/3 and 1/2. This implies that the Muskingum θ parameter which was thought of as to vary between zero and 1/2 will have to take a new references of range i.e. $1/3 \leq \theta \leq 1/2$.

5.0 Stablity Condition

Overton and Meadow [5] showed that for numerical stability the grid spacing $\Delta X/\Delta t$ should exceed, equal or be less than a critical value. Constantinides [6] illustrated that for any ΔX, the accuracy of a kinematic difference formulation decreases with increasing $\Delta X/\Delta t$, i.e. as Δt decreases. This was ascribed to numerical diffusion where by the extremes of a wave extend at a rate $\Delta X/\Delta t$ irrespective of the true dx/dt. For any $\Delta X/\Delta t$ the accuracy also drops off as ΔX increases since the grid becomes coarse. This dependence of accuracy on grid spacing must be taken into account when comparing different numerical schemes [7], [8].

5.1 Stability Condition of The Formulation based on The Weighted Residual Method

As observed from Weinmann et. al. [9] and Tsunematsu [10], care should be taken regarding all the values of the coefficients of discharge during the execution of calculations. As eq. (25) is a similar form of eq. (12), the coefficients of discharge of eq. (25) will not be able to take arbitrary values. From a hydraulic and mathematical point of view eq. (25) will be meaningful only if the coefficients of discharge are greater than or equal to zero. Therefore, the following condition must be satisfied.

$$\frac{1+2\alpha}{3\beta(1+\alpha)} \leq \frac{\Delta t}{K} \leq \frac{2+\alpha}{3(1-\beta)(1+\alpha)} \tag{28}$$

Considering the whole reach, if an equal time increment and a variable space increment is used, then for ß = 1/2 the following condition is obtained.

$$\frac{1+2\alpha}{2+\alpha} \leq \eta \tag{29}$$

Where

$$\eta=\frac{\Delta x_{min}\, Q_{min}^{1-P}}{\Delta x_{max}\, Q_{max}^{1-P}} \tag{30}$$

It could be seen from eq. (30) that as the maximum π value could take is unity, the maximum α value π could take is one. This is in agreement with the earlier assumption that the upper limit α of is unity. It could also be noticed from eq. (29) that if the discharge does not change much $\pi = 1$, then the value α of should be near unity, θ or θ should be near 1/2 and when there is a large variation in discharge $\pi = 1/2$, then the value of α should be closer to zero or θ should be closer to 1/3.

5.2 Stability Condition of The Muskingum - Cunge Method

The finite difference scheme (7) is stable for $0 \le \theta \le 1/2$ and becomes unstable for $\theta > 1/2$. It could be seen that discharge and flood wave attenuation increases as θ decreases, reaching a maximum for $\theta = 0$ and becoming zero for $\theta = 1/2$.

Anyhow for eq. (12) to be meaningful from an hydraulic point of view, c_1, c_2 and c_3 should be greater than or equal to zero and $\Delta X/\Delta t$ should be grater than or equal to the celerity. Considering this the limits of Δt in which eq. (12) would remain stable even for a variable space increment would be as follows.

$$2\theta \frac{\Delta x_{max}}{C_{min}} \le \Delta t \le \frac{\Delta x_{min}}{C_{max}} \qquad (31)$$

Where Δx_{max} and $x\Delta_{min}$ are the maximum and minimum space increment and C_{max} and C_{min} are the maximum and minimum celerity.

Calculating the above equation results in eq. (32)

$$\theta \le \frac{\Delta x_{min} Q_{min}^{1-P}}{\Delta x_{max} Q_{max}^{1-P} + \Delta x_{max} Q_{max}^{1-P}} \qquad (32)$$

If $\Delta x_{min} = \Delta x_{max}$, then eq. (32) could be written as eq. (33)

$$\theta \le \frac{1}{2}\left(\frac{Q_{min}}{Q_{max}}\right)^{1-P} \qquad (33)$$

For the finite difference scheme (7) to hold good, must be greater or equal to zero. Therefore, eq. (24) will give eq. (34)

$$Q_0 < F Q^{\gamma}$$

$$F = \frac{i\, b \Delta x}{k P} \quad \text{and} \quad \gamma = 1 - P \qquad (34)$$

As it could be seen form eq. (34) eq. (12) holds good only for a particular ratio of the discharge to the reference flow. Moreover for a complete valid solution the value of θ obtained from eq. (24) must satisfy the criterion eq. (32). These conditions make it quite difficult to obtain a value for θ for all practical circumstances.

As it is clear from eqs. (32) and (34) that the value of θ holds good only for a certain ratio of discharge to reference flow, for a given channel, a numerical example is stated below to prove the point.

For a wide rectangular channel as the hydraulic radius could be expressed as the water depth the following equations hold good.

$$k = b^{2/5}\left(\frac{n}{i^{1/2}}\right)^{3/5} \quad \text{and} \quad P = \frac{3}{5} \qquad (35)$$

Calculation were executed assuming a channel type of i = 0.005, b = 25 m and n = 0.03, and taking ΔX as 50 m. The maximum deviation of the space increment was taken as 10% from the standard value (i.e. ΔX_{min}=45 m a nd ΔX_{max} =55m). Fig - 2 represents the graphs calculated using eq. (24) and (32) for reference flow values Q_0 as stated in the graphs.

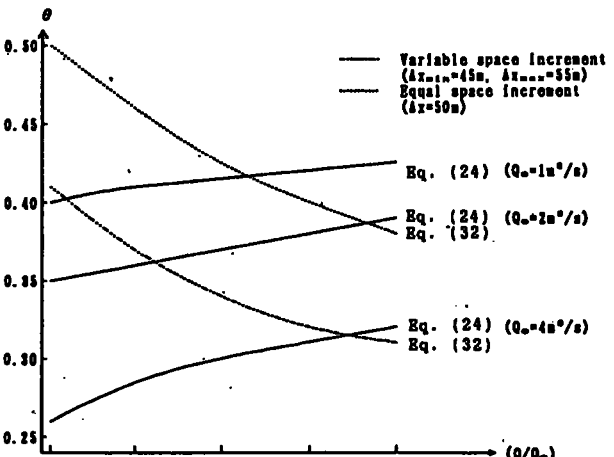


Fig 2 Graph of numerical example of the stability condition.

It could be seen from Fig - 2 that as the reference flow decreases the value of θ obtained from the parabolic differential equation increases and there by can take a value larger than the value necessary for the stability of the Muskingum - Cunge equation. This also happens as the ratio of discharge to reference flow increases. It could also be noted that if $\ominus$ is found from the second order approximation to diffusion routing, it will not necessarily yield values so that the Muskingum - Cunge equation will be stable. Also as the variation of space increment increases the range that $\ominus$ can take becomes smaller.

Therefore, it could be said that $\ominus$ obtained from the parabolic differential equation will yield values so that the Muskingum - Cunge equation will be stable only for high reference flows and low ratio of discharge to reference flow.

6.0 0Conclusion

The Muskingum - Cunge scheme is investigated and it is found that as the reference flow decreases, the following takes place. The weighting coefficient of the finite difference representation of the time derivative, found from the second order approximation to diffusion routing, will not necessarily yield values so that the conventional Muskingum - Cunge equation will be stable. This also happens as the ratio of discharge to reference flow decreases, implying that the weighting coefficient in the finite difference representation of the time derivative found from the second order approximation to the diffusion routing holds good as long as the variations are mild.

It is also found that as the standard deviation of the space increment increases the weighting coefficient of the finite difference representation of the time derivative derived from this equation will not necessarily yield values so that it will satisfy the stability condition of the Muskingum - Cunge equation. It could be also noticed that for the stability of this equation

$\Delta X_{min} Q_{min}^{1p} / \Delta X_{max} Q_{max}^{1p}$ must be greater than or equal to 1/2.

By investigating the weighted residual method and the Muskingum - Cunge method, it is found that the better equation to use would be eq.(27) and that the weighting coefficient of the finite difference representation of the time derivative must be between 1/3 and 1/2. It is also found that when there is no appreciable variation in discharge, the value that this coefficient must take should be closer to the upper limit 1/2, and when there is a large variation in discharge then the value that this coefficient must take should be closer to the lower limit 1/3.

7.0 Reference

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