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A METHODOLOGY FOR PARAMETER IDENTIFICATION OF GROUNDWATER AQUIFERS

by

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Abstract:

Inverse problem methodology has been used for geophysical prospecting purposes as well as in design to synthesize devices from given performance criteria. It has been also used to identify the location, material and value of unknown sources within an inaccessible region, using exterior measurements and geometric differentiation of finite element matrices. In this paper the same technique is used for identification of aquifer properties of anisotropic media and geometric boundary using hydraulic head measurements. The object function is defined in such a manner that it vanishes at its minimum when hydraulic head at certain observation wells matches the hydraulic head variation resulting from a set of assumed values of parameters such as aquifer properties and geometric boundary between two aquifer media. The method is demonstrated by identifying principal directions and transmissivities of two aquifer media at increasing complexity. This methodology is also demonstrated by identifying the boundary between two anisotropic aquifer media through geometric differentiation of finite element matrices.

1.0 Introduction

Groundwater flow models are important tools in the assessment and development of water resources as it is used for prediction of head and velocity of groundwater flow. Impacts of increased pumping in region or of groundwater recharge below irrigated agriculture, are some examples where an understanding and prediction of groundwater head and flow is required. Further, groundwater flow are a crucial component of all analysis of contaminant transport as the groundwater velocity field needs to be defined. The saturated flow through porous geologic environment have been extensively investigated and are quite well understood in the context of a wide variety of problems. The nature of these parameters appearing in the various forms of the fully saturated groundwave flow equation is reasonably well understood but the nature of these parameters when used to predict average flows over large distances through heterogeneous geologic deposits is poorly understood. Theoretical studies have explored the relationship between log scale transmissivity and the variability of local transmissivity, but our understanding is limited. The unpredictability of flow parameter is a major sources of uncertainty in groundwater modeling today. In most applications of groundwater flow models, parameter values are estimated through calibration using some type of inverse technique along with the use of well test estimates and geologic knowledge. Parameter values are chosen in such a manner that the prediction of observed head at selected observation points is satisfactory. The most important problem in the application of groundwater flow models is the specification of parameter values such as characterization of the geologic environment. Direct measurement is at best costly. Complete characterization in any case is impossible because non-destructive measurement method are not available. However, when appropriate data are available, parameter identification through inverse problem methodology provides a practical way to characterize large-scale distribution of properties such as transmissivity from hydraulic head data A large amount of theoretical literature has grown up around the groundwater inverse problem (Yeh 1986). It is known from these studies that parameter values estimated in this way are non unique and are very sensitive to errors in measures head data. A number of automated techniques have been suggested for dealing with these problems and for automated techniques have been suggested for dealing with these problems and for quantifying the resulting uncertainties. There is however no agreement on this approach to this problem. Automated techniques remain experimental in practice and most calibrations proceed by trial and error fitting procedures with no quantification of uncertainty (Committee on groundwater Modeling Assessment 1990). Parameter estimation remains one of the crucial challenges in successful groundwater flow modeling.

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1.1 Inverse Problem Methodology

A formal methodology for solving inverse problem has now evolved. Gitsusastro et al (1989) have taken a more general approach at the cost of computational complexity, the methodology that seeks to meet the design goal of electromagnetic devices, when it is not precisely attainable. In this process, we define an object function for given performance criterion, that would at its minimum satisfy the criterion. The generic device is described in terms of parameters and then the object function is minimized with respect to the parameters. At the minimum of the object function, the values of the parameters describe the particular device that would meet the design requirements. This is an iterative procedure, where series of equations each corresponding to an improved design are analyzed until the ideal is attained. Oristaglio et al (1985) is one of the earliest in inverting electromanetic fields by making a mathematical model of the interior. They used surface measurements of the externally induced electromagnetic fields to predict the profile of interior conductivity for geophysical prospecting purposes using a least squares objective function.

1.2 Regional groundwater Model

Two-dimensional confined flow in porous media is described by Bear (1979). In practice we encounter two dimensional confined flow in a region of anisotropic but homogeneous material. If the coordinate axes X and Y coincide with the principal axes of the hydraulic conductivity tensor, the governing partial differential equation is given by:

$$\frac{\partial}{\partial x}(K_{xx}\frac{\partial H}{\partial x} + \frac{\partial}{\partial y}(K_{yy}\frac{\partial H}{\partial y} = S_s\frac{\partial H}{\partial t} - q \qquad (1)$$

where H-Hydraulic head; K_{xx}, K_{yy} - Hydraulic conductivities in x,y directions respectively; S_s - Specific Storage; q- Net injection rate per unit volume. The dimensionality of the eq. (1) is often reduced by vertical integration because the vertical flow component is relatively unimportant. Thus we obtain

$$T_{xx}\frac{\partial^2 H}{\partial x^2} = T_{yy}\frac{\partial^2 H}{\partial y^2} = S_s\frac{\partial H}{\partial t} - q^* \qquad (2)$$

where $T_{xx} = K_{xx}$. b; $T_{yy} = K_{yy}$ b;S = b. S_s ; b - the aquifer thickness; q* - Net injection rate per unit area. A steady state analysis of groundwater aquifer is often made as a first approximation in parameter estimation. In some problems it will suffice to obtain average values of such parameters and the boundary conditions are averaged over a given period of time. Thus steady regional groundwater flow could be modelled by Poisson's equation given by:

$$T_{xx}\frac{\partial^2}{\partial x^2} + T_{yy}\frac{H\,\partial^2 H}{\partial y^2} = -q^* \qquad (3)$$

The boundary conditions consists of (i) $H = H_b (s_1)$ on fixed head boundary S_1 and (ii)

$$T_{xx}\frac{\partial H}{\partial x} + T_{yy}\frac{\partial H}{\partial y} + q^1 = 0$$

on S_2, where $1_x\, 1_y$ - direction cosines normal to the surface s_2; q' - flow rate per unit width of the boundary s_2. At the impermeable boundary q' = 0. In this paper the inverse problem methodology is demonstrated in identifying aquifer properties of anisotropic media, for the case of confined aquifer with hydro-geologic boundaries.

2.0 Inverse Problem Methodology and Finite Elements

The simulation of groundwater system that we commonly deal with is the direct problem where all the system parameters are assumed to be known and the system behaviour is simulated by sophisticated numerical analysis techniques. In reality we are often faced with the inverse problem, where the groundwater model parameters are to be estimated using the response of the system at few observation wells, as the physical structure of the aquifer cannot be observed directly. The problem of identification of aquifer parameters where the properties of inaccessible aquifers are to be estimated could be solved by inverse problem methodology. The work of Oristaglio et al (1985) is one of the earliest in inverting electromagnetic fields by making a mathematical model of the interior. They used surface measurements of the externally induced electromagnetic fields to predict the profile of interior conductivity for geophysical prospecting purposes using a least squares objective function. Thus our object function could be written as

$$F = \frac{1}{2}[H_c(i) - H_m(i)]^2 \qquad (4)$$

where H is the hydraulic head and n is the number of points at which H could be measured. The computed value of the measurement point i, H_c (i) is that given by the present estimate of groundwater system parameters and the measured value H_m (i) is what the actual response of the groundwater system. It is evident that when the unknown aquifer parameters are properly estimated, the object function which is otherwise positive would vanish. Thus using the simplest method namely the steepest gradient method, at every iteration moving towards the proper estimate of parameters, we must compute the gradient of the object function dF/dp -α (dF (j)/dp$_i$) gives us the change required in pi to take us towards proper value of the system parameters. Once the direction of optimization - (dF(j)/dp) is found, the problem now is to find the optimum step α along this direction. Several methods have been proposed but most commonly used is the quadratic or cubic approximation method. This is done by line search [Vanderplatt 1980] where for obtaining cubic approximation of object function F, four values of α, which envelope the optimal step α_{opt} are chosen. Then from the cubic approximation of F we find $\alpha_{opt}(k)$ which converges to the actual minimum. Although there are many schemes for minimizing the object function [Pironneau 1984], the gradient methods yield the fewest iterations, but the computation of the gradient with respect to every parameter by virtual displacement of the parameter can be inordinately expensive, slow and subject to numerical error. This cumbersome computation at every iteration, held back for long the use of gradient methods in solving inverse problems. This has been changed by recent innovations in using the finite element method (Oristaglio et al 1980, Gitsusastro et al 1989).

Let us consider the object function as a function of parameters p and hydraulic head distribution H in the aquifer given by F = F (p, H). Thus

$$\frac{dF}{dp} = \frac{\partial F}{\partial p} + \frac{\partial F}{\partial (H)}\frac{\partial (H)}{\partial p} = \frac{\partial F}{\partial p} + \frac{\partial F}{\partial H_1}+\frac{\partial H_1}{\partial p} + \frac{\partial F}{\partial H_2}\frac{\partial H_2}{\partial p} + \ldots \frac{\partial F}{\partial H_n}\frac{\partial H_n}{\partial p} \tag{5}$$

The term (∂F/ ∂p) is easily computed from the known form of the object function and for our case of steady confined flow (eq.3) it vanishes since no parameter explicitly enters it. The term (∂F/ ∂H) is also easily computed from the known form of F and is given by:

$$\frac{\partial F}{\partial H} = \begin{bmatrix} \frac{\partial F}{\partial H_1} \\ \frac{\partial F}{\partial H_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial F}{\partial H_n} \end{bmatrix} = [H_c(i) - H_m(i)] \; (6) \begin{bmatrix} \frac{\partial H_c(i)}{\partial H_1} \\ \frac{\partial H_c(i)}{\partial H_2} \\ \cdot \\ \cdot \\ \frac{\partial H_c(i)}{\partial H_n} \end{bmatrix}$$

where $\frac{\partial Hc(i)}{\partial Hj} = \left\{ \begin{array}{l} 1 \text{ if } i=j \\ 0 \text{ if } i=j \end{array} \right\}$

The computation of (∂F/ ∂H) is required and computation of this term has been prohibitively expensive [Pironneau 1984], but the finite element method gives us an ideal way of differentiating the unknown H without having to resort to a second solution with the parameters slightly perturbed. A second perturbed solution could be truly prohibitive [Pironneau 1984] particularly in the case of multi-parameter problems such as ours. Not only that, perturbations for the purpose of computing derivatives is also subject to numerical error. Oristaglio et al [1980] recognized the suitability of finite elements in differentiation fairly early, where this was employed in getting derivatives with respect to the material value of subsurface soil. This approach is based on the fact that the finite element matrices are expressed in terms of nodal coordinates and material and source values. Although this method of differentiation is analitycally exact as stated by Oristaglio et al [1980], it is also an approximation in so far as the finite element trial functions are approximate. Nonetheless as indicated by many applications [Coulomb 1983, Hoole et al 1985, 1991], the accuracy is far superior to that by perturbation. Though the computation of the gradient (dH/dp) is relatively simple, when using finite element, the alternate finite difference and boundary element formulations can pose serious problems (Hoole et al 1991]. Let us consider the eq. (3) which describes regional groundwater flow in homogenious anisotropic media to illustrate how finite elements help in the computation of the gradient of hydraulic head with respect to a parameter pi. A finite element discretization yields a matrix equation of the form

$$[P]. [H] = [Q] \tag{7}$$

The local contribution to P and Q respectively Pe and Qe for a finite element formulation employing quadratic triangular elements are given by:

$$Pe = [a_x.s_1 + b_x.\, s_2 + c_x\, .\, s_3]\frac{T_{xx}}{4A_r} + [A_y.S_1 + b_y.S_2 + C_y.S_3]\frac{T_{yy}}{4A_r} \tag{8}$$

$$Q_e = \frac{A_r.q}{12} \tag{9}$$

where the coefficients a_x., b_x., c_x. ay,by,cy and matrices S_1,S_2,S_3 are given in the appendix. The advantage of using such representation is that the coefficients are functions of coordinates of the corners of triangular element, no matter linear, quadratic or cubic triangular elements are employed, but only corresponding basic element matrices Si should be used [Schwarz 1988]. Thus it is seen clearly that P is differentiable with respect to the coordinates and transmissivity of the aquifer and Q is differentiable with respect to the coordinates and source q [Hoole et al 1991]. Therefore solving for H and differentiating the eq. 7 with respect to a parameter pi we obtain:

$$P\frac{\partial H}{\partial P_i} = \frac{\partial Q}{\partial P_i} - \frac{\partial P}{\partial p_i}.\, H \tag{10}$$

Which may be solved for ($\partial H/\partial p_i$) as H is already known from eq. 7 and ($\partial Q/\partial p_i$) and ($\partial P/\partial p_i$) are computable.

3.0 Application in Identifying Aquifer Parameters.

A hypothetical confined aquifer with hydro- geologic boundaries is used for simulation and then for demonstration of the inverse problem methodology. The description of the groundwater system consisting of two zones of different aquifer charateristics with a common boundary is shown in figure 1. The inverse problem methodology is employed to identify the system parameters at increasing complexity to demonstrate the performance of the methodology with increasing number of parameters. Initially two parameters namely the angles of principal axis in both zones θ_1, θ_2 are to identified assuming that the transsivities are known. Secondly four transmissivity values T_{xx} (1), T_{yy}(1), T_{xx} (2) and T_{yy} (2) are identified assuming that the angles are known. Thirdly all these six parameters are identified employing this methodology. Finally the methodology is used to identify a geometric parameter namely the boundary between the two zones as well as the transsivities T_{xx} (1), T_{yy} (1), T_{xx} (2) and T_{yy} (2).

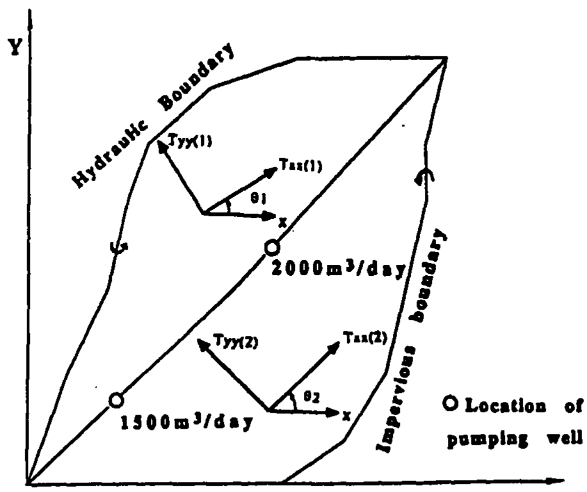

FIG. 1. Plan View of Hypothetical Aquifer System.

3.1 Identifying the angles of Principal directions

One way to deal aquifers which are homogeneous but anisotropic having principal axes incline to global coordinate axes X, Y at an angle ø, is to consider the local axes x, y to coincide with the principal axes. Them the coordinates with respect to principal directions are obtained by a simple transformation given by

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix}$$

Where ø is the parameter to be identified in each zone. The derivatives of P_e and Q_e (see equations 8 & 9) with respect to the angle ø are given by

$$\frac{\partial P_e}{\partial \phi} = \frac{T_{xx}}{4A_r}[S_1\frac{\partial a_x}{\partial \phi} + \frac{\partial b_x}{\partial \phi}S_2\frac{\partial c_x}{\partial \phi} + \frac{T_{yy}}{4A_r}S_3] + [S_1\frac{\partial a_y}{\partial \phi} + \frac{\partial b_y}{\partial \phi}S_2 + \frac{\partial c_y}{\partial \phi}S_3] \quad (11a)$$

$$\frac{\partial Q_e}{\partial \phi} = \frac{A_r . q}{12} \quad (11b)$$

The derivatives of coefficients a_x., b_x., c_x, a_y, b_y, c_y with respect to ø are given in the appendix for the convenience of the reader. The variation of object function with angles ϕ_1, ϕ_2 is shown in figure 2 which dispels any doubt of non-convergence for certain staring values of ϕ_1, and ϕ_2. The methodology is applied with starting values of ϕ_1, and ϕ_2 as 5 and the actual values are arrived at within five iterations as shown in figure 3.

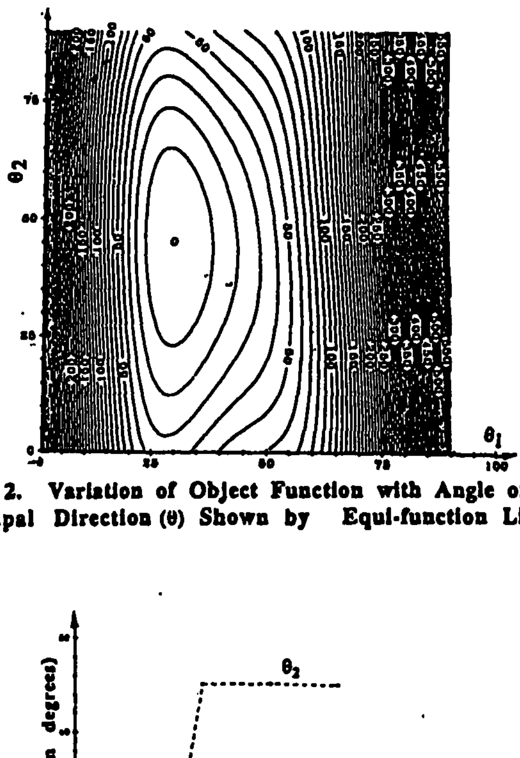

FIG. 2. Variation of Object Function with Angle of Principal Direction (θ) Shown by Equi-function Lines.

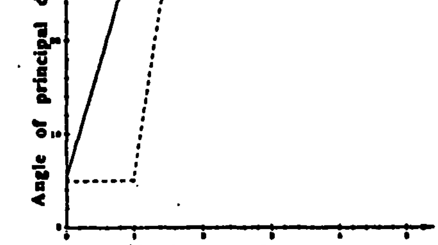

FIG.3. Convergence of Parameters θ1 and θ2

3.2 Identifying Transmissivities in the Aquifer Zones

When T_{xx} T_{yy} are the parameters to be identified, the derivatives of P_e and Q_e with respect to T_{xx} and T_{yy} are to be simply given by:

$$\frac{\partial P_e}{\partial T_{xx}} = [a_x.S_1+b_x.S_2 + C_x.S_3]\,\frac{1}{4A_r} \quad (12a)$$

$$\frac{\partial P_e}{\partial T_{yy}} = [a_y.S_1 + b_y.S_2 + C_y.S_3]\,\frac{1}{4.A_r} \quad (12b)$$

$$\frac{\partial Q_e}{\partial T_{xx}} = 0 \quad \cdot(12c)$$

$$\frac{\partial Q_e}{\partial T_{yy}} = 0 \quad (12d)$$

The variation of the object function with transmissivities in one zone is shown in figure 4. The methodology outlined above is used to identify the four parameters simultaneously with low initial values as shown in figure 5. The actual values of the transmissivities are obtained within 50 iteration, though beyond 25 iterations, the convergence is rather slow.

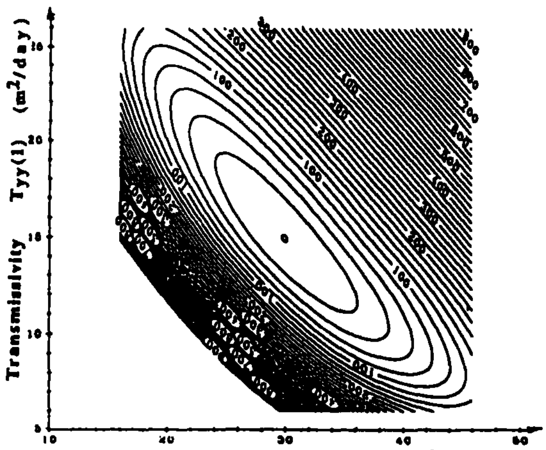

FIG. 4. Variation of Object Function with Transmissivities Txx(1), and Tyy(1) Shown by Equi-function Lines

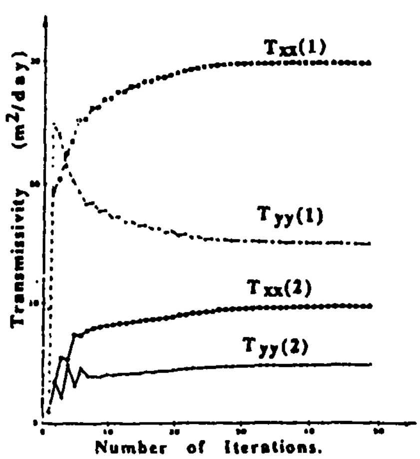

FIG.5. Convergence of Parameters Txx(1), Tyy(1), Txx(2) and Tyy(2).

3.3 Identifying Transmissivities and angle of principal directions

All six parameters were to be identifies simultaneously. In order to avoid large fluctuations in $\varnothing_1$, and $\varnothing_2$ initially, first few iterations were carried out keeping $\varnothing_1$, and $\varnothing_2$ fixed. Fast convergence was observed when corrections in the angle and transmissivity are done at alternative iterations rather than simultaneously. As shown in figure 6, the true values of the parameters are obtained after 100 iterations.

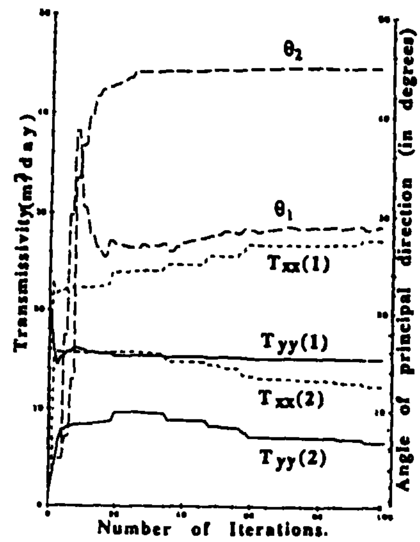

FIG.6. Convergence of Parameters Txx(1), Tyy(1) Txx(2), Tyy(2), θ_1 and θ_2 .

3.4 Identifying the Boundary and Transmissivities

The boundary between the two zones is approximated by a quadratic polynomial given by (see figure 7.):

$$y = K_o = k_1\, x = k_2\, .\, x^2 \quad (13)$$

where k_o, k_1 and k_2 are constants. The curve goes through points $a(x_a,y_a)$ $B(x_b,y_b)$, and $P(x_p,y_p)$. For convenience x_p is chosen to be fixed with a value of $(x_a.=x_b)$ /2 so that only y_p has to be evaluated to identify the boundary. In practice, from the field data an approximate range of y_p is known, When geometric parameter P_g is to be identified, the derivatives of P_e with respect to P_g may be given by:

$$\frac{dP_e}{dP_g} = \frac{\partial P_e}{\partial x_1}\cdot\frac{\partial x_1}{\partial P_g} + \frac{\partial P_e}{\partial y_1}\cdot\frac{\partial y_1}{\partial P_g} \quad (14)$$

When remeshing is done x coordinates of the nodal points lying on the boundary are kept fixed. thus $(\partial F/\partial X_1)$ turns out to be zero along these boundary nodes. Therefore when Y_p is the only geometric parameter $(\partial P_e/\partial y_p)$ is given by: $(\partial P_e/\partial y_p) = (\partial P_e/\partial y_1)\,(\partial y_1/\partial y_p)$. $(\partial P_e/\partial y_1)$ could be written as:

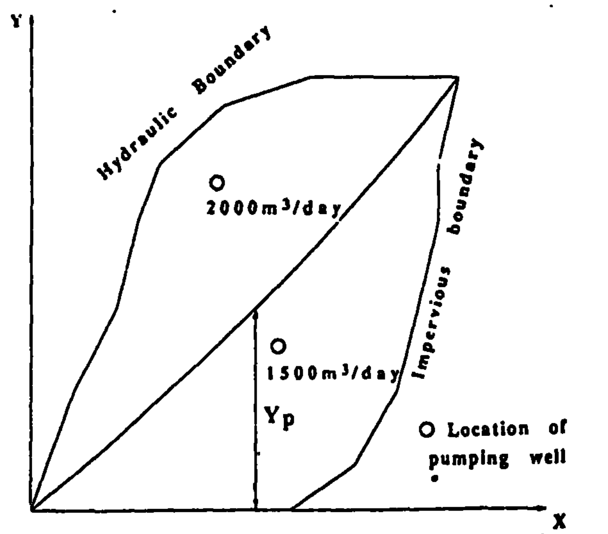

FIG. 7. Plan View of Hypothetical Aquifer System with a geometric parameter.

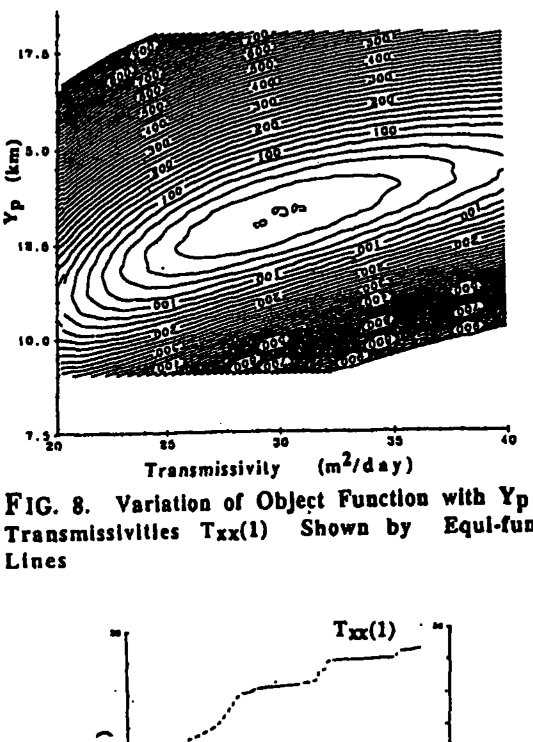

FIG. 8. Variation of Object Function with Yp and Transmissivities Txx(1) Shown by Equi-function Lines

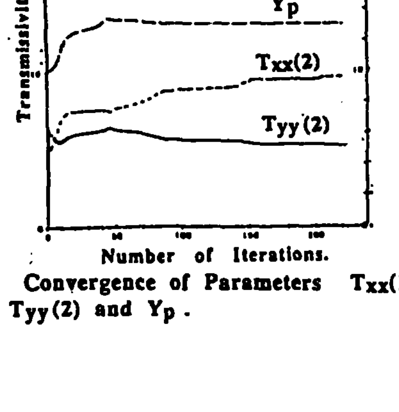

FIG.9. Convergence of Parameters Txx(1), Tyy(1), Txx(2), Tyy(2) and Yp.

$$\frac{\partial P_e}{\partial y_1} = \frac{T_{xx}}{4.A_r}\left[S_1 \frac{\partial a_x}{\partial y_1} + S_2 \frac{\partial b_x}{\partial y_1} = S_3 \frac{\partial c_x}{\partial y_1}\right] \tag{15}$$

Since the zonal boundary is approximated by a quadratic polynomial, for any nodal point (x_1, y_1) lying on the boundary APB the derivative (dy_1/dy_p) could be obtained as: $(dy_1/dy_p) = \{(x_1-x_2)(x_1-x_b)(x_p-x_a)(x_p-x_b)$. Thus (dP_e/dy_p) could be calculated easily. The methodology is applied to identify the boundary and the transmissivities of the aquifer zones. After each iteration only the elements around the boundary APB are remeshed . The variation of the object function with Y_p and one of the transmissivity is shown in Figure 8 to see whether it exhibits any local minima. Figure 9 shows that though the boundary parameter y_p converges around 100 iterations, the overall convergence takes place after 200 iterations.

4.0 Conclusions

A method based on the inverse problem technique used in design synthesis has been presented for identifying the properties such as transmissivity and principal direction of anisotropic confined aquifer using hydraulic head measurements. This methodology is demonstrated using simulated data of a hypothetical aquifer with two aquifer zones, surrounded by hydro - geologic boundaries. As the object function behaves well without any local minima in all the cases considered the convergence of the parameters to be identified is possible without any difficulty. As the number of parameters to be identified rapidly increases, the number of iterations required for convergence increases rapidly requiring well over hundred iterations for five or six parameters. Geometric differentiation of finite element matrices is used in the case of identifying zone boundary, where only the elements around zone boundary are remeshed at each iteration. Thus in the case of steady regional groundwater flow problems this methodology is very useful in identifying aquifer parameters. This methodology could be applied to the confined aquifer in the Vanathavillu basin which spreads over 40 square kilometers. In the case of unconfined aquifers this method may be used for regions where head variations are small compared with the depth of the aquifer. This may be the case in some regional aquifers in the dry and intermediate zones in Sri Lanka.

5.0 Acknowledgements.

The author wishes to express his sincere thanks to Prof. S.R. Hoole and the Dept . of Engineering, Harvey Mudd College, Claremont, California for providing all the assistance for this study

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Appendix I:

Coefficients of Triangular elements and Matrices S1 S2 S3

$$ax = \frac{(y3 - y1)2}{2Ar}$$

$$bx = \frac{(y3-y1) - (y2-y1)}{2Ar}$$

$$cx = \frac{(y2-y1)2}{2Ar}$$

$$ay = \frac{(x3 - x1)2}{2.Ar}$$

$$ay = \frac{(x3-x1)2}{2.\ Ar}$$

$$by = \frac{(x3-x1).\ (x2-x1)}{2Ar}$$

$$cy = \frac{(x2-x1)2}{2Ar}$$

$$S_1 = \frac{1}{6}\begin{bmatrix} 3 & 1 & 0 & -4 & 0 & 0 \\ 1 & 3 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -4 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 0 & -8 & 8 \end{bmatrix}$$

$$S_2 = \frac{1}{6}\begin{bmatrix} 6 & 1 & 1 & -4 & 0 & -4 \\ 1 & 0 & 1 & -4 & 4 & 0 \\ 1 & -1 & 0 & 0 & 4 & -4 \\ -4 & -4 & 0 & 8 & -8 & 8 \\ 0 & 4 & 4 & -8 & 8 & -8 \\ -4 & 0 & -4 & 8 & -8 & 8 \end{bmatrix}$$

$$S_3 = \frac{1}{6}\begin{bmatrix} 3 & 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & -4 \\ 0 & 0 & 0 & 8 & -8 & 0 \\ 0 & 0 & 0 & -8 & 8 & 0 \\ -4 & 0 & -4 & 0 & 0 & 8 \end{bmatrix}$$

where (x1,y1), (x2,y2) and (x3,y3) are the coordinates of the corners of the triangular element.

Appendix II:

Derivatives of the Local Finite Element Matrices With Respect to .

$$\frac{\partial a_x}{\partial ø} = \frac{-2\,(y3-y1)\,(x3-x1)}{Ar}$$

$$\frac{\partial b_x}{\partial ø} = \frac{(y3-y1)\,(x2-x1)=(x3-x1)\,(y2-y1)}{Ar}$$

$$\frac{\partial c_x}{\partial ø} = \frac{-2\,(y2-y1)\,(x2-x1)}{Ar}$$

$$\frac{\partial a_y}{\partial ø} = \frac{2(x3-x1)(y3-y1)}{Ar}$$

$$\frac{\partial by}{\partial Q} = \frac{(y3-y1)(x2-x1)+(x3-x1)(y2-y1)}{Ar}$$

$$\frac{\partial cy}{\partial Q} = \frac{2(y2-y1)\,(x2-x1)}{Ar}$$

Appendix III :

Notation

The following symbols are used in this paper

a_x	=	Coefficient of triangular element
b_x	=	Coefficient of triangular element
c_x	=	Coefficient of triangular element
a_y	=	Coefficient of triangular element
b_y	=	Coefficient of triangular element
c_y	=	Coefficient of triangular element
F	=	Object function
H	=	Hydraulic Head
H_b	=	Hydraulic head on Hydaulic boundary.
H_c	=	Computed hydraulic head
H_m	=	Measured hydraulic head
K_i	=	Coefficients of quadratic polynomial (i=0,1,2).
K_{xx}	=	Hydraulic Conductivity in x direction
K_{yy}	=	Hydraulic Conductivity in y direction
$1_x{}'\,1_y$	=	Direction cosines of normal to boundary.
pi	=	Parameter.
p	=	Finite Element Matrix
P_e	=	Local contribution of p.
Q	=	Vector
Q_e	=	Local contribution of Q.
q	=	Net injection-extraction rate.
q	=	Water seeping out of the boundary.
s_1, s_2	=	Hydro-geologic boundaries.
S_i	=	Basic element matries
T_{xx}	=	Transmissivity in x direction
T_{yy}	=	Transmissivity in y direction
x,y	=	local coordinate axes
x_i, y_1	=	Corrdinates of triangular elements' corners'.
X,Y	=	Global coordinate axes
$\dot{\alpha}$	=	a step
α_{opt}	=	Optimal step
θi	=	Direction of principal axis